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SOME DEVELOPMENTS ON NAVIER-STOKES EQUATIONS IN THE SECOND HALF OF THE 20th CENTURY

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Dedicated with deep respect to the memory of Jean Leray.

INTRODUCTION

The theory of Navier-Stokes equations (NSE) constitutes a central problem in contemporary mathematical physics. These equations are a physically well accepted model for the description of very common phenomena, and much effort has been devoted to them by fluid mechanic engineers, meteorologists, mathematicians and others; nevertheless many problems are still at the frontier of science. From the physical viewpoint some problems which are amazingly simple in their formulation are still unresolved, such as the determination of the thermal properties of turbulent flows or the description of the forces exerted by a turbulent flow on its boundary (e.g. airplane or pipe). On the mathematical side NSE are a model for the study of nonlinear phenomena and nonlinear equations which is itself in its infancy: NSE and the related Euler equations encompass four central problems in nonlinear equations, namely, well-posedness, oscillations, discontinuities, and nonlinear dynamics.

The mathematical theory of the NSE is rather technical and already very large despite being still incomplete. Hence it was not obvious what an article addressing such a vast subject should contain: description of results (without writing a catalogue of results), evolution of techniques and ideas, relations with turbulence and physics. It has not been possible to address satisfactorily all these questions in the present notes. There are certainly many important omissions, many important subjects to which not enough space has been devoted; choices have been made and undoubtedly others in the fields would have made different choices. A large list of references compensates in part for these

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defficiencies. Despite all their shortcomings I still hope that these notes can be useful to some readers; I myself have learned new things while writing them and I corrected or consolidated things I knew.

The name Navier-Stokes equations will apply here to both incompressible and compressible flow equations, although some authors use the name NSE for the incompressible flow equations only. Both cases will be addressed in this article but the emphasis will be on incompressible flows for which the mathematical theory is more advanced.

The mathematical theory of the NSE started with the pioneering work of J. Leray (1933, 1934a,b), who on this occasion introduced for the first time the concept of weak formulation of partial differential equations before the development of the distribution theory by L. Schwartz (1950, 1951), and shortly before S.L. Sobolev (1936) systematically introduced the spaces which bear his name. J. Leray has laid the basis of the mathematical theory of the incompressible NSE as we know it and he has introduced many tools and ideas used constantly since then. In fact, despite all the efforts, the progress has been relatively slow since the work of J. Leray. A beautiful description of Jean Leray's contributions to the theory of partial differential equations (including NSE) written by Peter Lax will appear in the collected works of Jean Leray (1998).

Concomitant to, but independent of the work of Leray is the work of Gunther, Lichtenstein, and Wolibner on Euler equations, which has been rediscovered long after. We are not aware of any other rigorous result during the 1930s and 1940s. However important empirical and heuristical results were derived, e.g. by Lamb, Prandtl, G.I. Taylor, von Karman and others. Also in the 1940s, A.N. Kolmogorov (1941) published his fundamental work on turbulence and J. von Neumann and his collaborators generated a new activity on the computational side and in meteorology. The theoretical study of the NSE using modern functional analysis resumes at the very beginning of the period under consideration with the well-known article of E. Hopf (1951). It then continues actively all along the second half of the century. The mathematical theory for compressible flows developed at the end of this period.

In this article we will emphasize two aspects of the mathematical theory: well-posedness, i.e. existence, uniqueness and regularity of the solutions in various function spaces in the incompressible case and the connection with turbulence, in particular the connection of the mathematical theory of NSE with the conventional theories of turbulence of Kolmogorov and Kraichnan. Other subjects are mentioned in varying detail including the compressible NSE, the Euler equations, optimal control, related equations corresponding to the coupling of fluid mechanics with other phenomena. The subjects essentially or totally untouched include the transition to turbulence in relation with bifurcation, the relations of the NSE with kinetic theory or with models of turbulence such as the $k - \varepsilon$ model, non Newtonian flows, multifluids and numerical approximation. As we said the problem of the numerical solution of the NSE equations was initialized by von Neumann and his collaborators in the 1940s. With the considerable development of the power of computers during the past decades, numerical simulation has developed as a subject of its own, Computational Fluid Dynamics, at the interface with engineering and physics; nevertheless this subject still raises problems of significant mathematical substance.

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Contents

PART I: THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

1. EXISTENCE, UNIQUENESS AND REGULARITY OF SOLUTIONS

The Navier-Stokes equations

It is useful to recall briefly the derivation of the Navier-Stokes equations (NSE).

We consider the motion of a fluid which occupies at time t a domain Ω_t of the space $\mathbb{R}^3$; we will assume that $\Omega_t = \Omega$ is independent of time since the mathematical difficulties for moving domains tend to hide the difficulties specific of the NSE. In fluid mechanics, the Lagrangian representation of the motion consists in providing the trajectory of each particle of fluid, $x = \Phi(a,t)$, where a is the position at time 0 of the particle, x its position at time t. The NSE in their most common form correspond to the Eulerian representation of the flow, which provides the vector fields $u = u(x,t)$ corresponding to the velocity of the particle of fluid which is at x at time t; for the notations, $x = (x_1, x_2, x_3), a = (a_1, a_2, a_3), u = (u_1, u_2, u_3)$. We have

$$u(x,t) = \frac{\partial \Phi}{\partial t}(a,t),$$

and conversely we can recover the Lagrangian representation of the motion from the Eulerian one by solving the systems of ordinary differential equations $(x_a(t) = \Phi(a,t))$:

$$\frac{dx_a(t)}{dt} = u(x_a(t),t), \quad x_a(0) = a. \tag{1}$$

The conservation of momentum equations read

$$\rho\gamma_i = f_i + \sigma_{ij,j},$$

where $\rho = \rho(x,t)$ is the density, $\gamma = \gamma(x,t)$ is the acceleration, $f = f(x,t)$ represents external volume forces applied to the fluid and $\sigma = (\sigma_{ij}(x,t))_{ij}$ is the Cauchy stress tensor; we have used the Einstein convention of summation of repeated indices and $\varphi_{,j} = \partial\varphi/\partial x_j$. We know from kinematics (chain rule differentiation) that

$$\gamma_i = \frac{\partial u_i}{\partial t} + u_j u_{i,j}.$$

On the other hand, for the so-called Newtonian fluids, the Cauchy stress tensor is taken as

$$\sigma_{ij} = 2\mu D_{ij}(u) + \lambda \text{ div } u \;\; \delta_{ij} - p \;\; \delta_{ij},$$
$$D_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where $p = p(x,t)$ is the pressure and $\mu > 0, \lambda \in \mathbb{R}$. Hence the general NSE read

$$\rho(\frac{\partial u_i}{\partial t} + u_j u_{i,j}) - \mu\Delta u_i - (\lambda + \mu)(\text{div } u)_{,i} + p_{,i} = f_i. \tag{2}$$

Another fundamental equation expresses the conservation of mass (continuity equation):

$$\frac{\partial \rho}{\partial t} + \text{ div } (\rho u) = 0. \tag{3}$$

Now if the fluid is incompressible, the volume of any part remains constant during the motion, which is expressed by

$$\text{div } u = 0. \tag{4}$$

In this case the conservation of mass equation implies that ρ is constant along the trajectories of the fluid. Hence if the fluid is homogeneous, $\rho(x,0) = \rho_0 > 0$ is independent of x and, dividing equation (2) by ρ_0, we obtain the NSE equations of incompressible homogeneous flows consisting of (4) and

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} - \nu\Delta u_i + p_{,i} = f_i, \tag{5}$$

or in vector form

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f. \tag{5$'$}$$

We have set $\nu = \mu/\rho_0 =$ the kinematic viscosity, and we have renamed p/ρ_0 and f/ρ_0 as p and f. The equations of incompressible nonhomogeneous flows consists of (2), (3) and (4) with simplifications in (2) and (3) resulting from (4). In all these equations viscosity is present, $\mu, \nu > 0$; the case $\mu = \nu = 0$ corresponds to inviscid or "perfect" fluids, in which case we recover the Euler equations.

The rest of Part I is devoted to equations (4), (5) and to simplified forms of physical and/or mathematical interest, namely stationary flows (u and p are independent of t), and linearized equations for which the quadratic term is dropped.

A remarkable property of the Navier-Stokes equations is that they are one of the very few (if not the only) *nonlinear equations* in mathematical physics for which the nonlinearity is derived from mathematical argument (just chain rule differentiation) and not from physical modelling.

Boundary and Initial Conditions

We supplement (4) and (5) with boundary and initial conditions. We assume that Ω is a bounded open set of $\mathbb{R}^3$ with a $\mathcal{C}^r$ boundary Γ, Ω lying locally on one side of $\Gamma (r \geq 2$ at least). The Dirichlet (or no-slip) boundary condition

$$u = 0 \quad \text{on } \Gamma, \tag{6}$$

(or $u(x,t) = g(x,t)$ given) corresponds to the case where the boundary Γ is materialized (solid) and at rest (or moving with prescribed velocity g). Another case of mathematical interest is the space periodic case, where

$$\begin{aligned} &u \text{ and } p \text{ are periodic with period} \\ &L_i \text{ in the direction } x_i, \;\; i = 1,2,3; \end{aligned} \tag{6'}$$

in this case Ω is the period $\overset{3}{\underset{i=1}{\Pi}} (0, L_i)$. In this case we also specify $\int_\Omega u = \int_\Omega f = 0$. Of course such a flow is not physically feasible. Nevertheless space periodic flows are of interest in the study of homogeneous turbulence; another physical difference between (6) and (6') is that the boundary condition (6) leads to the appearance of boundary layers near Γ when ν is small, which is the case for common fluids (air, water). From the mathematical viewpoint, and at our present level of understanding of the mathematical theory, there is not much difference between the boundary conditions (6) and (6') e.g. for well-posedness and long-time behavior. There are however some functional analysis difficulties with (6) - now well understood - which are easily resolved in the space periodic case with the use of Fourier series (see the section Function Spaces, Stokes Problem).

Other boundary conditions which will not be emphasized here correspond to Ω unbounded, or to channel flows (periodicity in two directions, Dirichlet condition in the third direction) and the "free boundary" condition such as that at the horizontal surface of a liquid, yielding boundary conditions of the Neumann type

$$u \cdot n = 0, \;\; (\sigma \cdot n)_\tau = 0 \quad \text{on } \Gamma,$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal on Γ and $(\sigma \cdot n)_\tau$ is the tangential component of $\sigma \cdot n (\sigma$ the Cauchy stress tensor). Due to (4), the latter condition reduces to

$$n \times \; \text{curl}\, u = 0 \quad \text{on } \Gamma.$$

We also supplement equations (4), (5), (6) or (6') with an initial condition for u,

$$u(x,0) = u_0(x) \;\; (\text{ given }), \;\; x \in \Omega. \tag{7}$$

The unknown functions u and p play very different roles; this will appear from the Leray (weak) formulation of the boundary and initial value problem (4)-(7), but we can see readily that, at each instant of time, including $t = 0$, p is a function of u expressed through the solution of a Neumann problem; indeed taking the divergence of (5'), we find

$$\Delta p = \text{ div } f - u_{j,i}u_{i,j}. \tag{8}$$

In the space periodic case (6'), this equation provides p as a function of u and f; in the case of (6), we take then the scalar product of (5') with n and obtain the Neumann boundary condition corresponding to this equation

$$\frac{\partial p}{\partial n} = f \cdot n + \nu \Delta u \cdot n \quad \text{on } \Gamma. \tag{9}$$

Function Spaces. Stokes Problem

We concentrate on (6), the modifications are easy for (6') and most of what follows is straightforward in this case. In the context of the L^2 space theory, we consider the space of test functions

$$\mathcal{V} = \left\{ v \in \mathcal{C}_0^\infty(\Omega)^d, \text{ div } v = 0 \right\},$$

$\mathcal{C}_0^\infty(\Omega)$ denoting the space of $\mathcal{C}^\infty$ functions with a compact support in Ω, and the closures of $\mathcal{V}$ in $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, hereafter denoted H and V; d is the space dimension, all the results in this section extending to the case where Ω is an open set in $\mathbb{R}^d$. The Sobolev space $H^m(\Omega)$ is the space of functions in $L^2(\Omega)$ whose distributional derivatives up to order m are in $L^2(\Omega)$; $H_0^1(\Omega)$ is the subspace of functions in $H^1(\Omega)$ vanishing on Γ. By virtue of the Poincaré inequality it is known that $H_0^1(\Omega)^d$, as well as V, are Hilbert spaces for the scalar product and the norm

$$((u,v)) = \Sigma_{j=1}^d \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right), \quad ||u|| = \{((u,u))\}^{1/2}$$

with

$$(f,g) = \int_\Omega f(x)g(x)dx, \quad |f| = (f,f)^{1/2},$$

u, v, f, g scalars or vectors; H is of course a Hilbert space for the norm $|\cdot|$. The spaces H and V are characterized as follows:

$$H = \left\{ v \in L^2(\Omega)^d, \text{ div } v = 0, v \cdot n|_{\partial\Omega} = 0 \right\},$$
$$V = \left\{ v \in H_0^1(\Omega)^d, \text{ div } v = 0 \right\}.$$

Further properties of H, in relation with the first cohomology space of Ω, appear in Foias and Temam (1978).

The Stokes problem is the stationary linearized version of (4), (5), (6) (or (6')), i.e. with $\nu = 1$,

$$\begin{aligned} -\Delta u + \text{ grad } p = f \quad &\text{in } \Omega, \\ \text{div } u = 0 \quad &\text{in } \Omega, \\ u = 0 \quad &\text{on } \Gamma. \end{aligned} \tag{10}$$

In the space periodic case the solution of this problem is easy using Fourier series. In the Dirichlet case the existence and uniqueness of u follows immediately from the weak formulation below using the Riesz representation theorem (or the projection theorem). More involved is the regularity theory of the Stokes problem which was developed for L^2 and L^p spaces, by Cattabriga (1961) and Solonnikov (1964, 1966), using the methods of the Agmon, Douglis and Nirenberg (1959, 1964) theory of regularity of elliptic systems. For L^2 spaces a simpler proof appears in Ghidaglia (1986). For f in $H^m(\Omega)^d, m \geq -1, u$ is in $H^{m+2}(\Omega)^3$ and p in $H^{m+1}(\Omega)$, with a linear continuous dependence of u and p on f; here Γ is assumed to be $\mathcal{C}^r, r \geq m+2$.

We set $D(A) = V \cap H^2(\Omega)^d$ and, for $u \in D(A), Au = -P\Delta u$ where P is the orthogonal projector in $L^2(\Omega)^d$ onto H (which we propose to call the Leray—Helmholtz projector). By the regularity results above, A is an isomorphism from $D(A)$ onto H; its inverse A^{-1} is self-adjoint positive and compact in H, with a complete orthonormal basis in H of eigenvectors:

$$Aw_j = \lambda_j w_j, \quad j \geq 1, \quad \lambda_j \to \infty \text{ as } j \to \infty.$$

The behavior of the λ_j as $j \to \infty, \lambda_j \sim c(j)^{2/d}$ ($d =$ the space dimension, $= 3$ here) has been derived in Metivier (1978); it is the same as for the eigenvalues of the Laplace operator (see e.g. Courant and Hilbert (1953)). One can also define the powers A^s of $A, s \in \mathbb{R}$, with domain $D(A^s)$; A^r is an isomorphism from $D(A^{s+r})$ onto $D(A^s)$.

We finish with a few words about the weak formulation of the Stokes problem; it is obtained by multiplying the first equation (10) by $v \in V$ (or $\mathcal{V}$), and integrating over Ω :

$$u \in V \quad \text{and} \quad ((u,v)) = (f,v), \qquad \forall v \in V.$$

Existence and uniqueness of u follows from the Riesz theorem; recovering p then follows from the following characterization of grad $\mathcal{D}'(\Omega)$, where $\mathcal{D}'(\Omega)$ is the space of distributions on Ω :

$$M \in \text{ grad } \mathcal{D}'(\Omega) \quad \Longleftrightarrow \quad < M, \varphi >= 0, \ \forall \varphi \in \mathcal{V}. \tag{11}$$

This characterization can be inferred from the De Rham theory of currents; a simpler proof due to L. Tartar (1976), uses the characterization by J.L. Lions (in Magenes and Stampacchia (1958)) of $L^2(\Omega)$ as the space of distributions φ such that grad $\varphi \in H^{-1}(\Omega)^d$ ($H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$). Finally a simple self contained proof of (11) is due to X. Wang (1993).

Weak formulation. Existence and Uniqueness Results

The weak formulation of the Navier-Stokes equations introduced by J. Leray is obtained by multiplying (5') by a test function $v \in \mathcal{V}$ (or V) and integrating over Ω. We denote by $u(t)$ the function $\{x \in \Omega \to u(x,t)\}$, and we then look for $u(t) \in V$ for (almost) all $t > 0$, and such that, in the distribution sense on $(0,T)$ or $(0,\infty)$:

$$\begin{gathered}\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + b(u(t),u(t),v) = (f(t),v), \ \ \forall v \in V, \\ u(0) = u_0.\end{gathered} \tag{12}$$

Here, and whenever the corresponding integrals are defined,

$$b(u, v, w) = \Sigma_{i,j=1}^{d} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx.$$

In the context of the L^2-theory we usually call weak solutions of (12), the solutions which belong to $L^2((0,T);V)$ and $L^\infty((0,T);H)$, $\forall T > 0$, and strong solutions, those which belong to $L^2((0,T);D(A))$ and $L^\infty((0,T);V)$, $\forall T > 0$. The results of existence and uniqueness are different according to space dimension. The existence of solution is generally obtained by a constructive method: constructing an approximate solution (e.g. by Galerkin method, or finite differences in time and/or space), and then passing to the limit using a priori estimates on the solution; one can use also the Leray-Schauder fixed point theorem or simply the Banach fixed point theorem (which usually produces solutions on a "small" interval of time, $0 < t < T_*$). The essential point in any proof seems to be the derivation of a priori estimates; those depend on the evaluation of the term b, and there are indeed numerous estimates of b usually obtained by combining Holder's inequality with some other functional inequalities (Sobolev, Agmon, Ladyzhenskaya, Gagliardo-Nirenberg, interpolation).

Leray established the existence and uniqueness of regular solutions for all time in space dimension 2, for the whole space (1933) and then for some interval of time $(0, T_*), T_*$ depending on the data, for a bounded domain (1934a). He also considered the three dimensional case (1934b) and, for the flow in the whole space, he proved the existence and uniqueness of a regular solution on some interval $(0, T_*), T_*$ depending on the data, and the existence for all time of a weak solution discussing also (see below) the possible occurrence of singularities. Hopf (1951) proved the existence for all time of a weak solution for three dimensional flows in a bounded domain with zero velocity at the boundary. Hopf used the same framework as Leray, but he used the Galerkin method to construct approximate solutions, while Leray constructed approximate solutions using an approximate equation obtained by mollifying the nonlinear term. The next basic result is the uniqueness of weak solutions in space dimension two proved by Lions and Prodi (1958). At about the same time Ladyzhenskaya (1958, 1959) improved the existence result of Leray of strong solutions for two dimensional bounded domains; this result takes its final form when supplemented by results proven subsequently, in particular the regularity for the Stokes (and stationary Navier-Stokes) equations.

J. Leray called *turbulent* the weak solutions of the NSE. One of his motivations when he introduced the concept of weak solutions was to consider a class of solutions allowing the curl vector to become infinite for the description of turbulent flows: the problem that he raised of the possible occurrence of singularities in 3D turbulent flows is still unresolved, existence of singularities in the 3D NSE has not been proven nor disproven (see below).

In summary, in the context of L^2 spaces, the status of the existence-uniqueness theory is as follows:

- In space dimension $d = 2$, the theory is fairly satisfactory, the problem is well posed in the sense of Hadamard: existence and uniqueness of weak solutions, of strong solutions if the data are suitably regular; more generally the solution is as regular as allowed by the data (including $\mathcal{C}^\infty$ regularity and analyticity), provided the data

u_0, f, Ω are sufficiently regular; and we have continuous dependence on the data in the corresponding function spaces.

- In space dimension $d = 3$, we have only partial results: existence and uniqueness of a strong solution on some interval $(0, T_*), T_*$ depending on the data; existence of weak solutions on $(0, +\infty)$. Uniqueness of weak solutions is still an open problem, as well as the existence for all time of strong solutions. Of course, as in space dimension 2, the strong solutions, as long as they exist, are as smooth as allowed by the data, up to $\mathcal{C}^\infty$ regularity and analyticity; in fact there are no "intermediate levels" of regularity: as soon as the weak solution is e.g. in $L^6((0,T);V)$, or in $L^\infty(\Omega \times (0,T))^3$, it is a strong solution, as smooth as permitted by the data.

Remark 1.1

All the aforementioned results constitute by now the core of the classical L^2 theory of the NSE and, beside the original articles quoted above, they can be found in the books by Ladyzhenskaya (1969), J.L. Lions (1969), Constantin and Foias (1988), Temam (1984, 1995) (emphasis on the periodic case in the latter); a long (but not exhaustive) list of related results can be found in Marion and Temam (1997). The recent book by P.L. Lions (1996) contains the classical results and many new results. ■

Partial results are available in space dimension 3, and among them, the following: Leray (1933) showed that in the absence of forcing ($f = 0$), all solutions of NSE are eventually smooth (i.e. after some $T_* > 0$ depending on the data); it was shown in Serrin (1963) that if a strong solution exists on $(0,T)$, then there is no other weak solution on $(0,T)$. Foias, Guillopé and Temam (1981), Duff (1989, 1990, 1991) show that the weak solutions belong to $L^{2/3}(0,T;D(A))$, and to other spaces $L^q(0,T;H^m(\Omega)^d)$, $0 < q < 1, m > 2$.

We will recall hereafter a number of results concerning the possible occurrence of singularities or the blow-up of solutions; but now we conclude this section by a comment on a not too well-known result of Kato and Fujita (1962) showing that a smooth solution to the three-dimensional Navier-Stokes equations exists for all time if f is small in some sense and u_0 is small in $H^{1/2}(\Omega)^3$. With a slightly different presentation, and assuming that $f = 0$ for simplicity, the proof is based on the energy type equation obtained by replacing v by $A^{1/2}u(t)$ in (12); on properly estimating the nonlinear term we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A^{1/4}u|^2 + \nu|A^{3/4}u|^2 &= -b(u,u,A^{1/2}u)\\ &\leq c_1|A^{1/4}u||A^{3/4}u|^2\end{aligned}$$

Hence if $|A^{1/4}u_0| \sim |u_0|_{H^{1/2}} < \nu/c_1$, the norm $|A^{1/4}u(t)|$ decreases for all $t > 0$; a similar relation obtained by replacing v by $Au(t)$ in (12) implies then that the H^1 norm decays as well:

$$\frac{1}{2}\frac{d}{dt}||u||^2 + \nu|Au|^2 \leq c_1|A^{1/4}u||Au|^2.$$

If u_0 is concentrated on the high frequencies ($u_0 = \Sigma_{j=J}^{\infty} u_{0j}w_j, J$ large), then $||u_0|| \geq \lambda_J^{1/4}|A^{1/4}u_0|$. We conclude then that the solution to the three dimensional (3D) NSE exists and is smooth for all time, for arbitrary large values of $||u_0||$, (and similarly for $|f|$), if u_0 and f are highly oscillating; deep results implying similar conclusions were proven in the context of Besov spaces by Cannone and Meyer (1995); see also an announcement of Bondarevsky (1996), and see below the, result of Furioli, Lemarié-Rieusset and Terraneo (1997). This clearly shows that the existence of strong solutions to the 3D NSE depends not only on the magnitude of u_0 and f but also on their high frequency components (see Section 2); there is therefore need to use methodologies and/or function spaces which take into account the spectral properties of the data (and of the solution).

Remark 1.2

We will not speculate here on the possible occurrence of singularities in the 3D NSE, $||u(t)||$ becoming infinite in finite time; let us notice however that this would mean that "much activity" still occurs at small wavelengths, say smaller than the collision mean free path, or even the diameter of an atom; for example the amount of enstrophy ($||u(t)||^2 = |\text{ curl } u(t)|^2_{L^2(\Omega)}$) still concentrated on such wavelengths would be infinite. It would then be necessary to reconcile this fact with the physical foundations of the Navier-Stokes equations which use the hydrodynamics limit of kinetic theory for the definition of ν. Note that the same remark does not apply to the Euler equations.

Other results, other spaces ($L^p, \mathcal{C}^\infty, \mathcal{C}^\omega$, Hardy)

In the context of L^2 spaces, semigroup theory was used by Fujita and Kato (1964), see also Henry (1981), von Wahl (1985), Pazy (1983) and more recently Ben Artzi (1994), Brezis (1994). The emphasis above was on bounded domains but most of the previous results extend easily to the unbounded case for the evolution equation. Furthermore a large distinct literature exists for stationary unbounded domains with emphasis on exterior domains (i.e. complement of a compact smooth set); the problems here include the nature of the decay as $|x| \to \infty$ of $u(x)$ and, on the functional side, the fact that the natural space specified by energy relations, $\{u \in \mathcal{D}'(\Omega)^d, \partial u_i/\partial x_j \in L^2(\Omega), i,j = 1,\dots,d\}$ is not $H^1(\Omega)$ anymore. See in particular Finn (1959a,b, 1961, 1965 a,b), and the articles of Heywood (1972, 1974a,b, 1976) which contain also a study and an unexpected property of the space V for unbounded domains. The utilization of Sobolev spaces with weight in the study of the stationary and time dependent Navier-Stokes equations in unbounded domains is considered by Babin and Vishik (1990), Babin (1992) and others. In a series of articles, M. Schonbek (1991, 1992, 1995, 1996) studies the decay as $t \to \infty$ of the solutions to the NSE and related equations in the whole space $\mathbb{R}^2$ or $\mathbb{R}^3$, obtaining optimal algebraic decay rates for solutions with large data; the main idea for the decay is the method of Fourier splitting.

A number of authors have studied free boundary value problems attached to the Stokes and Navier-Stokes equations; see e.g. Abergel and Bona (1992), Beale (1981, 1984), Solonnikov (1977, 1982).

The L^p theory of the NSE includes a large number of results on existence and/or uniqueness in spaces $L^p(L^p)$ or in spaces $L^p(L^q)$ (i.e. $L^p(0,T;L^q(\Omega)^d)$). Such results were derived all along the period under consideration, and very early by Serrin (1959, 1962, 1963) and Prodi (1959). Solonnikov (1964, 1968) derived the existence and regularity theory for the linear Stokes evolution problem (i.e. (5') without the term $(u \cdot \nabla)u$); from this, one can derive many results for the NSE by interpolation and boot strapping (i.e. considering $(u \cdot \nabla)u$ as a source term). Many articles and the book of von Wahl (1985) emphasize the L^p theory. Although some L^p results can be recovered from the L^2 theory using in particular the work of Solonnikov, many results necessitate a totally different approach and rethinking and reworking several aspects of the theory, starting from the analog of the Helmholtz-Leray decomposition of vector fields ($L^2(\Omega)^d = H \oplus H^{\perp}$). Recently Furioli, Lemarié-Rieusset and Terraneo (1997), proved the uniqueness of Kato's mild solution $u \in \mathcal{C}([0,\infty); L^3(\mathbb{R}^3))$, the norm of u_0 in $L^3(\mathbb{R}^3)$ being small.

The L^2 or L^p regularity for the solutions of the Stokes or Navier-Stokes equations in non smooth domains has been investigated by Serre (1983) for piecewise regular domains (angles, corners) and for Lipschitz domains by Fabes, Kenig and Verchota (1988), Shen (1991), Brown and Shen (1995) among others. For piecewise $\mathcal{C}^2$ smooth domains (domains with corners or angles), see e.g. and Osborn (1976) and Grisvard (1985).

We return to smooth domains. The $\mathcal{C}^\infty$ regularity of strong solutions can be derived from the L^2 theory observing that, if Ω is $\mathcal{C}^\infty; \cap_{m=1}^{\infty} H^m(\Omega) = \mathcal{C}^\infty(\bar{\Omega})$; see Guillopé (1982, 1983) and other statements in Marion and Temam (1997). The space analyticity of solutions was proven by Masuda (1967). The time analyticity with values in $D(A)$ or other spaces was established by Iooss (1973) and by a different method by Foias and Temam (1979). The (space) regularity in Gevrey classes for the space periodic case was proven in Foias and Temam (1989); related results appear in Henshaw, Kreiss and Reyna (1990). In a series of articles, Grubb, Solonnikov and Geymonnat study the Stokes and Naiver-Stokes equations using pseudo differential operators (see e.g. Grubb and Solomikov (1992), Grubb (1995), Grubb and Geymonat (1977, 1979)).

Coifman, Lions, Meyer and Semmes (1993) studied the behavior of the inertial (nonlinear) term $(u \cdot \nabla)u$ in Hardy spaces, using compensated compactness (Tartar (1979)), and derived regularity results in Hardy and L^1 spaces. Cannone and Meyer (1995) (see also Cannone (1995)) studied the existence of solutions in Besov and other spaces, for $\mathbb{R}^2$ or $\mathbb{R}^3$, using the Littlewood-Paley decomposition.

The results in all these function spaces are similar to those in L^2 spaces, namely: existence and uniqueness of solutions on a small interval of time, and possibly for all time if $d = 2$ or under some condition if $d = 3$. All these approaches in various function spaces are not foreign to each other; on the contrary they are interrelated and benefit mutually from one another.

The behavior and regularity of the solutions of the NSE as $t \to 0$, is a problem in partial differential equations, which arises already in the context of the heat equation or of the evolution Stokes (linearized) equations: in the case of a bounded domain (e.g. boundary condition (6) (but not (6'')) the regular solutions blow up near $t = 0+$ in higher Sobolev norms ($H^m(\Omega), m \geq 3$), unless the data u_0 and f satisfy certain compatibility condition. The difference with e.g. the heat equation is that the compatibility conditions are global in space, due to the effect of the pressure; this question is studied in Iooss (1970) using semigroup theory, Heywood (1979), Heywood and Walsh (1994) and Temam (1982). This singular behavior may affect the numerical solution of the equations in bounded domains (see e.g. Gresho (1990)).

The backward uniqueness for the NSE was proved by Bardos and Tartar (1973) and by Ladyzhenskaya (1975): if $u = u_1$ and $u = u_2$ are regular solutions of the first equation

(12) for all $t < 0$ and if $u_1(0) = u_2(0)$, then $u_1(t) = u_2(t)$ for all $t < 0$.

A few other partial results on the well posedness of the NSE in space dimension 3: Fursikov (1980) proved that the 3D NSE are well posed (existence for all $t > 0$ of a strong solution) for all f in a dense set of $L^p(0, T; V')$, for any $p, 1 < p < 4/3$. Also a number of results were proven showing that the 3D NSE are well posed for 3D thin domains for $\Omega = \Omega_\varepsilon = \omega \times (0, \varepsilon), \omega \subset \mathbb{R}^2, 0 < \varepsilon < \varepsilon_0$, for a suitable ε_0, for classes of "large data" u_0, f; see Raugel and Sell (1992, 1993), and Temam and Ziane (1996); also Temam and Ziane (1997), obtain a similar result in a spherical domain $a < r < a + \varepsilon$, in relation with the study of geophysical flows.

A few words to finish about space dimension $d \geq 4$. For the nonlinear evolution equation (full NSE), the results are slightly less good but essentially similar to the 3D case; see e.g. Lions (1969). For the nonlinear stationary case, the regularity of the solutions has been recently investigated by Frehse and Ruzicka (1994a,b) and Struwe (1997); it uses the fact, observed in Serrin (1959), that $\frac{1}{2}|u|^2 + p$ satisfies a maximum principle (steady NSE).

Singularities, Self-similarity and Blow-up

Consider a weak solution u to the 3D NSE (12) where f and u_0 are assumed to be smooth (at least $f \in L^2(0, T; H), u_0 \in V$); if $u \in L^6(0, T; V)$, or if $u \in L^\infty(\Omega \times (0, T))^3$, then u is a strong solution. Hence if there exists a weak solution which is not strong, if singularities occur, then $||u(t)|| = |\text{ curl } u(t)|_{L^2(\Omega)^3}$ becomes infinite in finite time, or $|u|_{L^\infty(\Omega \times (0,T))} = +\infty$. In his pioneering work, Leray studied the possible occurrence of singularities and noticed that $\{t \in [0, T]; ||u(t)|| = +\infty\}$ has Lebesgue measure 0, and even a $\frac{1}{2}$-Hausdorff dimension 0 in $[0, T]$; furthermore the complement of this set in $[0, T]$ is a countable union of semi-closed intervals $[a_i, b_i)$. Scheffer (1977) was the first to study the size of the singular set in space and time; subsequently Caffarelli, Kohn and Nirenberg (1982) showed that the singular set $\{(x, t) \in \Omega \times (0, T), |u(x, t)| = +\infty\}$ has a zero one-dimensional Hausdorff dimension. In particular the singularities cannot lie on a smooth curve but at most on a "smaller set". Very recently simplified derivations of the results by Caffarelli, Kohn and Nirenberg have been given by Fang-Hua Lin and Chun Liu (1997) and by Gang Tian and Zhouping Xin (1997).

Leray also observed that if a regular solution to the 3D NSE on $[0, T)$ becomes singular at T, then $||u(t)||$ must blow up at least like *const* $/\sqrt{T-t}$ as t approaches $T - 0$. No such solutions have been found so far. Leray suggested that there may be singular selfsimilar solutions of the form

$$u(x, t) = \frac{1}{\sqrt{2a(T-t)}} U\left(\frac{x}{\sqrt{2a(T-t)}}\right), \tag{13}$$

where $a > 0$. He showed that if $U \not\equiv 0$ is a solution of the following system in R_y^3 :

$$-\nu \Delta U + aU + a(y \cdot \nabla)U + (U \cdot \nabla)U + \nabla P = 0, \text{ div } U = 0, \tag{14}$$

and if certain norms of U are finite (in particular $U \in L^\infty(\mathbb{R}_y^3) \cap L^2(\mathbb{R}_y^3)$) (Leray 1934, p. 225)), then the function u above develops a singularity at $t = T - 0$. Recently Necas, Ruzicka and Sverak (1996), proved that the only solution of (14) belonging to $L^3(R^3_y)$ is $U \equiv 0$. As for the stationary Navier-Stokes equations, $\frac{1}{2}|U|^2 + P$ satisfies a maximum principle, and this is used to prove the nonexistence of solution of (14). Even more recently Tai-Peng Tsai (1997) has shown that no similarity solution unless identically zero, has locally finite energy and locally finite enstrophy. These results preclude then the existence of physically acceptable singularities of the type (13).

With totally different motivations, Cannone and Meyer (1995), Cannone (1995), consider the existence of self similar solutions of the NSE in $\mathbb{R}^3 \times (0, \infty)$, of the form (13) with $T - t$ replaced by t; they formulate it as an initial value problem with an initial data homogeneous of degree -1, and solve this initial value problem for u_0 small in the appropriate space. Self similar solutions of other nonlinear equations have been extensively studied recently; see e.g. among many references Giga and Kohn (1985), Souplet and Weissler (1997).

2. ATTRACTORS AND TURBULENCE

In this section we discuss a few points concerning the relations between the NSE and Turbulence. This is again a vast subject by itself and this section is by no mean a thorough description of the corresponding literature. In particular we will not address the question of transition to turbulence which is related to stability and bifurcation theory, and we will concentrate on the permanent regime of turbulence and fully developed turbulence.

The first attempt at connecting NSE and turbulence was that of Leray, who conjectured the appearance of singularities in the NSE and introduced the concept of turbulent solutions. The work of A.N. Kolmogorov and the conventional theory of turbulence were then based on phenomenological concepts using in fact very little the NSE. The mathematical research program underlying the current research described below is: what can we learn about turbulence from the NSE? The attempts at giving some modest contributions to this question have consisted in rigorously proving certain well accepted facts in the conventional theory of turbulence, for which one can give a mathematical content and, on the other hand, on building some mathematical tools which are or seem suitable for further developing this connection.

There are mainly two roads for a mathematical description of turbulence; as we will see they are not totally unrelated, and they are also related to the NSE:

i) The conventional theory of turbulence is based on a statistical description of the turbulence. From the mathematical viewpoint the natural object is then a measure defined on the function space H (see below) which evolves with time according to the Navier-Stokes equations.

ii) The dynamical system approach. Following the developments in dynamical system theory and the ideas of S. Smale and D. Ruelle and F. Takens, the permanent turbulent regime is related to the global attractor which encompasses all the large time behaviors of the solutions.

Statistical solutions of the NSE

Statistical solutions of the NSE are commonly considered in the conventional theory of turbulence where the rapidly oscillating physical quantities are averaged in time and/or space. The corresponding measures μ are explicitly or implicitly introduced: close to the mathematical approach are e.g. the books and articles of Batchelor (1970), Onsager (1949), Orszag (1972), Monin and Yaglom (1971), Frisch (1995); see also the forthcoming book by Foias, Manley, Rosa and Temam (2000).

In the mathematical literature, Foias and Prodi developed the mathematical concept of statistical solutions to the NSE which appears in the long memoir of Foias (1973). The evolution of the probability measures μ_t on H governed by the NSE is given by the Hopf equation (1951). This is the Fourier transform of the Liouville equation

$$\frac{d}{dt}\int_H \Phi(u)d\mu_t(u) + \int_H \{((u,\Phi'(u)) + b(u,u,\Phi'(u)) - (f,\Phi'(u))\}\, d\mu_t(u) = 0,$$

where Φ runs over a suitable class of test functionals such that $\Phi'(u) \in V$. For this equation the existence and uniqueness theorems analogue to those of Leray (for 3D NSE), are proven in the reference above. Also by using the functional Fourier transform, similar results are proved for the Hopf equation. These results concern flows in bounded domains or in the periodic case. The case of homogeneous flows (spatially invariant measures) appropriate for the physically important case of homogeneous turbulence was treated by Vishik and Fursikov (1977a,b, 1980); they also investigated the relation with the moments of the measures. Subsequently self similar statistical were sought by Foias and Temam (1980, 1983), Foias, Manley and Temam (1988); their existence is related to a variant of the Leray equation (14) $\{a(y\cdot\nabla)U\}$ replaced by $\{-a(y\cdot\nabla)U]\}$.

Stationary Statistical Solutions

Time independent statistical solutions are important because of ergodicity and of their relations with the attractors.

In particular the following existence and approximation result has been proved: if u is solution of the NSE (5), (6) or (6'), (7), and if f is time independent, then the time averages of u

$$\frac{1}{T}\int_0^T u(\cdot,t)dt, \tag{15}$$

"converge," as $T\to\infty$, to a stationary statistical solution of the NSE, $\mu=\mu_f$. Ergodicity is conjectured in fluid mechanics and it is believed that time and space averages converge to the same limit so that this limit is uniquely defined. From the mathematical viewpoint the convergence of the averages holds for subsequences $T'\to\infty$ or in the weak sense of a Banach limit, LIM (see e.g. Dunford and Schwartz (1958) and applications in Foias and Temam (1980), Bercovici, Constantin, Foias and Manley (1995)). Typically, for any weakly continuous function Φ from H into $\mathbb{R}$,

$$\mathrm{LIM}_{T\to\infty}\frac{1}{T}\int_0^T \Phi(u(t))dt = \int \Phi(u)d\mu_f(u). \tag{16}$$

In the two dimensional case the measure μ_f is invariant under the semigroup $\{S(t) : u(0)\to u(t)\}_{t\geq 0}$, associated with (5), (6-6'),(7); invariance reads

$$\int \Phi(S(t)u)d\mu(u) = \int \Phi(u)d\mu(u), \quad \forall t\geq 0,$$

and a weaker form of invariance holds in space dimension three. This measure μ is a solution to the Hopf equation (a differential equation with infinitely many variables in the space H). Other properties of the stationary statistical solution μ_f are derived in the references quoted above. In particular we recall that such a measure is carried by the global attractor $\mathcal{A}$, i.e.

$$\mu(H \setminus \mathcal{A}) = 0. \tag{17}$$

Stochastic Navier Stokes equation

The stochastic NSE have also been extensively studied without direct reference to the conventional theory of turbulence. Let us quote in particular the case where the forcing f includes a white noise: Bensoussan and Temam (1973), Viot (1976), DaPrato and Zabczyk (1996). See also the very recent articles of Flandoli and Gatarek (1995) and Flandoli and Maslowski (1995); the former considers rather general white noises and looks for solutions which are martingales or stationary (in the probabilistic sense); the latter studies the uniqueness and the ergodicity of the invariant measure for an additive noise.

By studying the stochastic transport equation, Avellaneda and Majda (1991, 1992, 1994) have obtained precise results relevant to turbulence theory.

Attractors

The permanent regime of a turbulent flow is related to the long time behavior of the solutions of the NSE. In the dynamical system approach to turbulence we are interested in the global attractor of the equations which encompasses all the large time behaviors for all initial data u_0 (the forcing f being fixed and time independent). Besides establishing the existence of the global attractor which follows from general theorems on dynamical systems of infinite dimensions, the main contributions have been to show that the attractor has finite dimension, to estimate the dimension from below and from above and to give upper bounds on the dimension of the attractor which are physically relevant, i.e. which agree with related estimates derived in the conventional theory of turbulence.

The dynamical system generated by the NSE and the existence of the attractor appear in Ladyzhenskaya (1973, 1975). Foias and Temam (1979) proved that the global attractor for the 2D NSE (i.e. (5), (6) or (6') and (7) with f independent of time) has finite Hausdorff and fractal dimensions; the same is true in space dimension three for any functional invariant set X which is bounded in V $(S(t)X = X, \quad \forall t \geq 0)$. In the last article the dimension of the attractor $\mathcal{A}$ is exponential in terms of the Reynolds number or of the Grashof number, $Gr = |f|\nu^{-2}L^{3-d/2}$, L a typical length of Ω. Subsequently, using implicitly or explicitly the concept of Lyapunov exponents, bounds which are polynomial in Re or Gr are derived by Babin and Vishik (1983a, 1983b, 1985, 1986), Constantin and Foias (1985), Constantin, Foias and Temam (1985, 1988), Ladyzhenskaya (1985, 1987), Lieb (1984), Ruelle (1982/83), Temam (1986). The latter article gives a bound $\dim \mathcal{A} \leq cGr$ for large Gr as in space dimension two for boundary condition (6); CFT (1988) gives an upper bound $\dim \mathcal{A} \leq cGr^{2/3}(\log Gr)^{1/3}$ for the 2D NSE with boundary condition (6') and this bound is nearly optimal: Liu (1994) has proven that, in this case, dim $\mathcal{A} \geq cGr^{2/3}$.

Babin and Vishik (1985, 1986) were the first to derive a lower bound on the dimension of the global attractor for the NSE; they considered the 2D space period flow in an elongated domain $(0,1) \times (0,L)$ and found a bound, for large L, of the form dim $\mathcal{A} \geq cL$. Ziane (1997), improving the general results in Constantin, Foias and Temam (1985), obtains in this case an optimal upper bound dim $\mathcal{A} \leq c'L$, which matches the lower bound of Babin and Vishik.

The physical relevance of the estimates on the dimension of the attractors comes from the concept of finite dimensionality of turbulent flows which we discuss below. For more complete discussions on attractors for the NSE see e.g. the books by Babin and Vishik (1992), Hale (1988), Ladyzhenskaya (1991), Temam (1997) and the references therein.

The global attractor in 2D or a functional invariant set bounded in V in 3D is the union of complete orbits $\{u(t)\}_{t\in\mathbb{R}}$, which are solutions for all time of the first equation (12). It follows from Foias and Temam (1979) that these orbits are analytic in time in a band of the complex time plan $|Im\tau| < \delta$, with a width δ valid for all orbits on the attractor. The value of δ seems to have physical relevance and has been studied in a number of articles by Foias and co-authors; see e.g. Foias (1997).

It is desirable to have more informations about the attractors and the dynamics of the flow on the attractor, but, unfortunately, we know little beside these results.

Finite Dimensionality of Turbulent Flows

In the permanent regime, turbulent flows display a finite dimensionality which is emphasized in the book of Landau and Lifschitz (1953) describing the "finite number of degrees of freedom of turbulent flows." Consider for simplicity the space periodic case (6'): in 3D, by the Kolmogorov law, the spectrum of energy (for statistical averages)

$$E(k) = \frac{1}{2}\Sigma_{\substack{j\in\mathbb{Z}^d\\|j|\geq k}} |\hat{u}_j|^2 \quad \text{for } u = \Sigma_{j\in\mathbb{Z}^d}\hat{u}_j e^{ij\cdot x},$$

is very small and decays exponentially for $k > k_d$ where k_d is the Kolmogorov dissipation wave length, of the order of $k_0 Re^{3/4}$, where k_0 is a typical macroscopic length. Hence all active modes are statistically included in the ball $|j| \leq k_d$ and we easily count $(k_d/k_0)^3$ active modes. The attractor dimension was estimated by $c(k_d/k_0)^3$ in Constantin, Foias, Manley and Temam (1985), Constantin, Foias and Temam (1985), with a suitable definition of k_d. A similar result holds in 2D with the bound of the form $c(k_\eta/k_0)^2$, where k_η is the Kraichnan dissipation length.

Other concepts of finite dimensionality of flows have been introduced and studied: Foias and Prodi (1967) were the first to introduce the concept of determining modes: if $f(t) = f$ and u_1, u_2 are two solutions of (12) with initial data u_{01}, u_{02}, then if P_N is a suitable finite dimensional projector and if

$$P_N(u_1(t) - u_2(t)) \to 0 \quad \text{as } t \to \infty,$$

then

$$u_1(t) - u_2(t) \to 0 \quad \text{as } t \to \infty.$$

The concept of determining modes is extended by Foias, Manley, Temam and Treve (1983). The concept of determining nodes is introduced in Foias and Temam (1984) and estimates on the number of determining nodes comparable to that of the attractor are proven by Jones and Titi (1992).

Another useful concept of finite dimensionality of flow is the squeezing property of trajectories appearing in Foias and Temam (1979).

All these concepts of finite dimensionality give a mathematical meaning and a rigorous proof to the physical concept of “finite number of degrees of freedom” of turbulent flow.

Finally the concept of inertial manifold (IM) is another form of finite dimensionality of flows. The existence of IM has been proven for a number of dissipative equations including the NSE with hyper viscosity ($\epsilon(-\Delta)^r$ instead of $-\nu\Delta$) but not for the NSE themselves; see Foias, Sell and Temam (1985, 1988). The existence of an inertial form for the NSE themselves was announced by Kwak (1993), but the proof is not complete.

When it exists, an inertial manifold is a finite dimensional smooth manifold, to which all trajectories converge at an exponential rate; its equation of the form $z = \Phi(y)$ $(z = (I - P_N)u, y = P_N u$ where P_N is a suitable finite dimensional projector), gives a slaving law of the high modes by the low modes. As we said the existence of an IM for the NSE is an open problem but many forms of approximate IMs have been obtained (see e.g. Foias, Jolly, Kevrekidis, Sell and Titi (1988), Foias, Manley and Temam (1987, 1988), and a number of other references in Temam (1997)).

Other Connections between NSE and Turbulence

It is agreed in the conventional theory of turbulence that the time averaged energy dissipation rate is independent of the viscosity. Considering the flow in a channel Doering and Constantin (1992, 1995), have derived an upper bound on the rate of dissipation of energy which agrees with Kolmogorov scaling. X. Wang (1997) has extended their results to more general boundary driven flows: an essential technical tool here is the construction of a suitable extension inside the domain of a function defined on the boundary Γ of Ω, the construction of X. Wang improving a classical construction of Hopf. For the space periodic case see Foias (1997).

Finally Bercovici, Constantin, Foias and Manley (1995) further investigated the relations between the attractor of NSE and the corresponding statistical solutions.

As we said before, solutions of 2D NSE with initial conditions on the global attractor have the property of global time analyticity. The analyticity is on a strip of width δ in the complex time domain, where δ is independent of the orbit and is a decreasing function of the Grashof number. It was shown in BCFM (1995) that, consequently, the frequency spectrum, (i.e. the Fourier transform $P(\omega)$ of the two-time correlation of the turbulent velocity at a point in space; $P(\omega)$ is a positive measure) decays at high frequencies at least as fast as $exp(-\delta|\omega|)$. Here the frequency spectrum is given a rigorous definition which avoids the customary assumptions of the metric indecomposability of the phase space containing the turbulent solutions of 2D NSE.

PART II: OTHER PROBLEMS, OTHER EQUATIONS

3. COMPRESSIBLE and INVISCID FLOWS

Compressible viscous flows

Evolutionary Equations

As indicated before, the compressible Navier-Stokes equations (CNSE) are the equations (2) and (3). These equations arise in applications involving high Mach number flows of nondilute, compressible fluids. One should distinguish here between compressible fluids and compressible flows –a low Mach number flow of a compressible fluid will have nearly constant density, and therefore can reasonably be described by the incompressible Navier-Stokes equations (referred to as INSE in this paragraph). Another issue in the compressible case is the role of the pressure which is very different than in the incompressible case. In the later case p is a function of u (through the solution of the Neumann problem (8), (9)) and in most of the mathematical theory of the INSE it disappears and it is recovered at the end. In the compressible case, this is an independent function and by just counting naively equations and unknowns in (2), (3), we see that one more equation is needed. For barotropic flows, this supplementary equation is the state equation of the fluid, $P = P(\rho)$; for nonbarotropic flows, the equation of state is of the form $P = g(\rho, e)$, where e is the internal energy of the fluid and we supplement then (2), (3) with an equation for e, the energy equation (expressing conservation of energy, the first principle of thermodynamics). Here we consider only barotropic flows, and the corresponding CNSE.

Of course, many of the mathematical issues for CNSE will be the same as for INSE –existence, uniqueness, continuous dependence, and large time behavior. But there are two important respects in which the focus is somewhat different : in the applications involving CNSE, the pressure is by far the largest force, so that the viscosity and convection terms are less important. This means that a very large part of the mathematical analysis is involved with controlling the density and pressure pointwise, or at least in some appropriate norm; this issue just does not arise for the INSE. The other aspect, related to the first, is that the important qualitative features of solutions will be largely viscosity-independent ; that is, one hopes to understand solutions of the CNSE to some extent in terms of the canonical structures associated with the corresponding Euler equations of inviscid flow-shock waves, shear waves, rarefaction/compression waves, contact discontinuities.

Theorems concerning existence and uniqueness begin with the 1D theory of Kanel (1968) and Kazhikov and Shelukhin (1977), among many others; note that, unlike the case for INSE, the 1D theory is not trivial. Their analysis consists in nonstandard energy methods starting from the observation that, for local in time solutions, the total entropy is nondecreasing in time. This approach was extended to two and three dimensions in a series of papers of Matsumura and Nishida, starting in (1979). The analysis is much more technical, and is based to a greater extent on the correct prediction of asymptotic decay rates associated with the corresponding linearized equations. They prove the global existence of small (close to a constant state) smooth solutions with small, smooth initial data.

More recent results relax these restrictions on the initial data. First Lions (1993) then

Kazhikov and Weigant (1995) apply more modern techniques of weak compactness to obtain global solutions with *large* initial data in certain cases of barotropic flow ($P = P(\rho)$). Then Hoff (1995) analyzes in depth the effect of initial discontinuities for the full (nonbarotropic) CNSE, finding that singularities convect with the flow and decay exponentially in time, more rapidly for smaller viscosities and larger sound speeds ; this effect is obtained as a consequence of the *hyperbolicity* of the underlying Euler equations for inviscid flow (and is therefore one example of the point of view expressed above).

External forces do not appear in an important way in the above work. The question alluded to above, concerning the relationship between the solutions of the compressible Euler equations and the solutions of the CNSE is important, and a large amount of work is being invested in it. The only results so far are in 1D, and it is proven e.g. in Tai-Ping Liu and Yanni Zeng (1997), that a “viscous shock”, which is a traveling wave solution of the 1D CNSE having the physical attributes of an inviscid shock wave, is dynamically stable. This is the first result showing that the canonical structures associated with the inviscid compressible Euler equations have important significance for the solutions of the CNSE.

Steady Equations

The governing equations are (2) and (3) without the time derivatives; $\mu > 0$ and $3\lambda + 2\mu \geq 0$ is required by the second law of thermodynamics (in the evolutionary case as well). We assume that $p = g(\rho)$, usually $p = k\rho$ or $p = k\rho^\gamma, \gamma > 1$. The case where p depends also on the temperature necessitates the introduction of the heat (or energy) equation and will not be considered.

The stationary problem is usually considered in a smooth bounded domain Ω of $\mathbb{R}^d$ or, in the exterior case, in the complement of such a domain; $u = u_*$ is prescribed on $\partial\Omega, u_* \cdot n = 0$ in most articles, and $u \to u_\infty, \rho(x) \to 1$ as $|x| \to \infty$ in the exterior case.

Two types of results have been derived:

1) The existence and uniqueness of strong solutions near the equilibrium $\rho_0 = 1, u_* = u_\infty (u_\infty = 0$ by default in the interior case).

We have then a mixed elliptic-hyperbolic system for the unknowns $\sigma = p - 1$ and $v = u - u_\infty$ and this system is solved by a fixed point method using one of the following approaches:

- Elliptic regularization of the system and local estimates using specific techniques introduced by Matsumura and Nishida (1983); see e.g. Farwig (1989), Padula (1987), Pileczas and Zajaczkowski (1990), Valli (1987).
- Application of a Leray-Helmholtz decomposition of the vector field $v = w + \nabla\varphi$, where div $w = 0$ and writing the induced equations. See a series of results by Novotny (1994, 1997) and other articles to appear.

2) Another type of results is the existence of weak solutions.

First results are due to P.L. Lions (1993a,b) who established, among others, existence of weak solutions $(\rho, u) \in L^q(\Omega) \times W_0^{1,2}(\Omega)(1 < q(\gamma) < +\infty)$ for the insentropic case, under the conditions $\gamma > 1 (d = 2)$ and $\gamma \geq \frac{5}{3} (d = 3)$, e.g. in Ω a bounded domain with the above boundary conditions, as well as a partial regularity of these solutions: $\rho \in L^{\infty}_{\text{loc}}(\Omega), u \in W^{1,s}_{\text{loc}}(\Omega) \quad (\forall 1 < s < +\infty)$ and $\Pi = k\varrho^{\gamma} - (2\mu_1 + \mu_2) \operatorname{div} u \in W^{1,2}_{\text{loc}}(\Omega) \quad (\forall 1 < s < +\infty)$ provided $\gamma > 1$ $(d = 2), \gamma > 3$ $(d = 3)$. His proof relies on the notion of a renormalized solution for the continuity equation and on rewriting the momentum equation letting appear the commutator $\varrho \left\{ v_i \partial_i (-\Delta)^{-1} \partial_j (\varrho v_j) - \partial_i (-\Delta)^{-1} \partial_j (\varrho v_i v_j) \right\}$. The key point in the proof is the weak compactness of the above mentioned term obtained by using the results of Coifman, Lions, Meyer and Semmes (1993). The use of Helmholtz decomposition in the context of weak solutions and application of a div-curl lemma in Hardy spaces is discussed in Novotny (1996).

As indicated in the Introduction, the theory of the CNSE is more recent than that of INSE, but the evolution has been rapid in past years; also the problems and difficulties are different.

The recent book of P.L. Lions devoted to the INSE is followed by a second volume devoted to the CNSE, which just appeared.

Euler Equations

This paragraph gives a brief description of results and problems related to the Euler equations which could motivate a separate article. The interested readers can find substantial developments and discussions on the subject in the books of Majda (1984), Chemin (1995), and P.L. Lions (1996) and in the references therein.

The Euler equations describe the motion of nonviscous fluids, also called "inviscid" or "perfect". We write $\lambda = \mu = 0$ in (2) or $\nu = 0$ in (5), (5')[1] in this case the Cauchy stress tensor is spherical $\sigma_{ij} = -p \;\; \delta_{ij}$. The fluid can be again compressible or not, and in the later case, homogeneous or not. We emphasize the case of homogeneous incompressible flows; we will not mention incompressible nonhomogeneous flows and we will be very succinct for compressible inviscid flows: in fact in this case the mathematical theory of the Euler equations is closer to the theory of conservation law equations than to that of NSE.

Incompressible Flows

In the incompressible homogeneous case, the Euler equations are (5), (5') and (4); for an initial boundary value problem we prescribe the initial value of u as in (7) and the boundary condition can be space periodicity as in (6') or, instead of (6), we would just require nonpenetration:

$$u \cdot n = 0 \quad \text{on } \Gamma. \tag{18}$$

The pressure is again an auxiliary variable which can be expressed in terms of u, at each instant of time, by solving the Neumann problem (8), (9): of course $\nu = 0$ in (9), this changes considerably the functional dependence of p on u, and one can take advantage of

[1] We say a few words below about vanishing viscosities, i.e. μ or $\nu \to 0$.

this for proving the existence of solution (see below). Alternatively we have a variational formulation similar to (12) where we set $\nu = 0$ and require e.g. $v \in H \cap H^1(\Omega)^d$; hence the inertial (nonlinear) term is the essential term in this equation.

It is useful to write the equation for the vorticity $\omega = \operatorname{curl} u$, obtained by applying the curl operator to (5'); we find

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \operatorname{curl} f. \tag{19}$$

In space dimension $d = 2, \omega$ is a scalar, and the term $(\omega \cdot \nabla)u$ in the right-hand-side of (19) vanishes; hence (for $f = 0$), the vorticity is conserved along the streamlines. The case $d = 3$ is significantly different.

Concerning existence and uniqueness of solutions of the boundary value problem (5) (with $\nu = 0$), (7) and (b) (or (6'), or $u \to 0$ at infinity if $\Omega = \mathbb{R}^d$), here are a few of the known results:

(i) In all space dimensions, existence and uniqueness of a smooth solution on a small interval of time. More precisely let $X_m = H \cap H^m(\Omega)^d$; if $u_0 \in X_m$ and $f \in L^\infty(0, T; X_m), m - 1 > d/2$, then u exists and belongs to $\mathcal{C}([0, T_*); X_m)$, for some $T_* \leq T$ depending on u_0 and f (and Ω). The proof of existence is based on the a priori estimate

$$\left(\frac{d}{dt}\right) ||u||_{H^m} \leq c\,(||u||_{H^s} ||u||_{H^m} + ||f||_{H^m}) \tag{20}$$

valid for any $s > 1 + d/2$ $(c = c(s, \Omega))$.

(ii) In space dimension 2, existence and uniqueness of a smooth solution for all time $T > 0$: if $u_0 \in \mathcal{C}^\alpha(\bar{\Omega})$ is given, $\operatorname{div} u_0 = 0, u_0 \cdot n = 0$ on $\partial\Omega$ and if f is given in $\mathcal{C}^{1+\alpha,0}(\bar{\Omega} \times [0, T])$, then the solution of the Euler equations exists and is unique, $u, p, \partial u/\partial x_i, \partial p/\partial x_i \in \mathcal{C}(\bar{\Omega} \times [0, T])$ (p unique up to a function of t as usual).

(iii) In space dimension 2, $\Omega = \mathbb{R}^2$, if $f = 0$ and the initial vorticity ω_0 is a bounded *signed* measure compactly supported and $\omega_0 \in H^{-1}(\mathbb{R}^2)$ vortex sheet, then there exists for all $T > 0$, a solution $u \in L^\infty(0, T; L^2_{\text{loc}}(\mathbb{R}^2)^2)$ to the Euler equations.

The mathematical theory of Euler equations was initialized by Lichtenstein (1930) and Wolibner (1933), for $d = 2$; the proof of (ii) appears in Kato (1967). The proof of (i) is given in Kato (1972) for the whole space, and in Ebin and Marsden (1970), Bourguignon and Brezis (1974) and Temam (1975) (completed by Temam (1986)); for Ω bounded Temam (1975) gives a short proof based on an estimate of p in terms of u which could be useful elsewhere. The proof of (iii) is due to Delort (1991); the initial proof was substantially simplified by Schochet (1995) and the result was improved by Majda (1993).

It follows from (20) that the solution of the Euler equations remains smooth as long as $||u(t)||_{H^s}$ remains finite; a stronger result by Beale, Kato and Majda (1984) requires only the norm of $\operatorname{curl} u$ in $L^1(0, T_*; L^\infty(\Omega))$ to remain bounded; Ponce (1985) requires only the maximum norm (in space) of the deformation tensor $D_{ij}(u)$ to remain integrable in time.

More regularity in H^m spaces up to $\mathcal{C}^\infty$ regularity can be obtained as long as the solution exists, if u_0 and f are sufficiently regular. Note that, in contrast to the Navier-Stokes equations, there is no regularizing effect, $u(\cdot, t)$ is as smooth as $u_0(\cdot)$, no more.

The analyticity in time (in a suitable domain of the complex time plan) of the solutions of the Euler equations was proven by Bardos, Benachour and Zerner (1976). For the utilization of Besov spaces for existence and uniqueness of solutions of the Euler equations, see M. Vishik (1997a,b).

Arnold (1972) remarks in the context of the Euler equations that"there appear to be an infinitely great number of unstable configurations". The question of instability is closely tied to the structure of the spectrum of the linearised Euler operator which, unlike the case of Navier-Stokes, is non-elliptic. Friedlander and Vishik (1991,1992,1993) constructed a tool for detecting instabilities in the essential part of the spectrum based on a geometric quantity that can be viewed as a "fluid Lyapunov exponent".

As for the Navier-Stokes equations, an important question is the possible occurrence of singularities in finite time. The problem is open but, in the case of the Euler equations there are partial results and there have been more attempts to prove the occurrence of singularities or obtain "numerical evidence" of the existence of singularities (such as in Kerr (1997)), than the contrary. There is also a whole chapter in the theory of the Euler equation related to the case of nonsmooth solutions (vortex patches: the vorticity is a step function); see the references in the review article by Constantin (1995). In particular Chemin (1995) proved that if the initial data is a vortex patch in $\mathbb{R}^2$ (i.e. constant on a compact set) and the boundary of the support belongs to $\mathcal{C}^{1,\alpha}$ for some $\alpha, 0 < \alpha < 1$, then the solution exists for all time (as a vortex patch) and the boundary of the support remains in $\mathcal{C}^{1,\alpha}$ with the same α. Let us mention also the study of the Navier-Stokes or Euler equations with a measure as initial data. This problem has been investigated by a number of authors; see e.g. Cottet (1986), Giga, Miyakawa and Osada (1988), Michaux and Rakotoson (1993), Kato (1994), Constantin and Wu (1995). The results of Ben-Artzi (1994) and Brezis (1994) give the existence and uniqueness of a smooth solution for all time of the Navier Stokes and Euler equations with $f = 0$ and initial data in $L^1(\mathbb{R}^2)$ for NSE and in $L^1(\mathbb{R}^2) \cap L^r(\mathbb{R}^2) r > 2$, for the Euler equations.

Another important connection between the Euler and Navier-Stokes equations is the behavior of the NSE as $\nu \to 0$. This is well understood for the whole space or for the space periodic case as long as the solution to the Euler equations is smooth, see Kato (1984); the remaining issue in this case is the expression of the rate of convergence in term of ν. The problem of convergence of the solutions of the NSE to the Euler equations[2] is completely open and remains one of the major problems in mathematical physics if the solutions to the corresponding Euler equation is not smooth (even in the whole space) or in the case of a domain with boundaries, even if the solution to the corresponding Euler equation is smooth. Concerning the development of singularities DiPerna and Majda (1987) show, in a seminal work, the complexity of the phenomena which might occur: in particular concentrations and oscillations which produce a loss of kinetic energy, a phenomena which hampers the existence theory for the Euler equations. For bounded domains, the problem includes that of turbulent boundary layer which is open on the mathematical side (see some remarks below). Some activity has recently appeared: Sammartino and Caflish (1996) proved the convergence for the full NSE, for a small interval of time in the half-plane in the context of analytic solutions using an

[2]Some authors even believe that the solutions to the NSE do not converge to the solutions of the corresponding Euler equations.

abstract form of the Cauchy-Kowalewska theorem; Sammartino (1996) also studied the asymptotic behavior of the time dependent Stokes problem at vanishing viscosity in the half plane. R. Temam and X. Wang (1997) proved the convergence of the solutions of the NSE to that of the Euler equations for a bounded domain in the 2D case, making explicit the boundary layer function, by assuming a physically reasonable assumption: the tangential gradient of the pressure is bounded at the boundary; or the tangential derivative of one of the velocity components does not grow too fast in the boundary layer as the viscosity approaches zero. Kato (1984) proved a similar result assuming that the whole gradient of the velocity does not grow too fast.

The classical Prandtl equation, which is presumed to be, to some extent, a valid approximation of the NSE in the boundary layer have been studied by Oleinik (1963) and Fife (1967). Recently Weinan E and Engquist (1997) proved the blowing up of smooth solutions of the unsteady Prandtl equation for certain compactly supported data.

Compressible Fluids

As we said, Euler equations for compressible flows belong essentially to the theory of conservation law equations and they relate to the NSE in the sense that some authors believe that a good understanding of the way the Euler equations approximate the NSE (in the whole space) is needed to advance the theory of compressible NSE.

Here are briefly a few results; a thorough description of which, up to 1972, is given in Lax (1973). The global existence of a weak solutions for small data in the barotropic and nonbarotropic cases follows from Glimm's theorem ((1965)) which is valid for general conservation laws. Di Perna (1983a,b), applying for the first time compensated compactness to systems of conservation laws, studied the existence of solutions of barotropic flows. Following the work of Di Perna, Gui-Qiang Chen (1996, 1997) using as well compensated compactness proved, for such flows the global existence of solutions, where initial data and solution may include vacuum ($\rho = 0$). The theorem of Glimm and Lax (1970) applies to barotropic flows, for solutions and initial data in L^∞, with $u(t) \in BV$, for $t > 0$, even if this is not true at $t = 0$, but assumes that the oscillations of u_0 are of small amplitude. See also Nishida and Smoller (1973), which gives a global weak solution for $P = \rho$ and with limitations on the total variation of u_0 if $P = \rho^\gamma, \gamma > 1$. Serre (1997), Grassin and Serre (1997) obtain under appropriate assumptions the existence of smooth solutions defined for all time in any space dimension.

4. SOME OTHER PROBLEMS AND EQUATIONS

Related and Approximate Equations

Here we mainly list some related equations leading to very similar mathematical problems and some approximate equations generating in general specific mathematical problems.

The Navier-Stokes (or Euler) equations can be coupled to other equations when other physical phenomena are present. For fluids conductor of electricity (e.g. plasmas, seawater, or cryolite in the industry of aluminum), electric and magnetic phenomena are present; the governing equations are then the magnetohydrodynamic equations consisting of a proper coupling of the Maxwell and NSE equations. For thermohydraulics, NSE are coupled with the heat equation, in particular in the context of Boussinesq approximation. For combustion or reacting flows, NSE are coupled with the heat equation and more or less complex chemistry equations; the chemistry equations include the equations for the concentration for each species. In all these cases, viscosity can be or not present in the fluid equation and in the other equations, and the fluid can be compressible or not. In the case of incompressible homogeneous Newtonian fluids, viscosity being present in all equations, the theory of well-posedness of the equations leads to exactly the same results as for the corresponding NSE alone; maximum principles apply to the temperature and the concentrations.

Modified forms of the NSE have been introduced for shallow waters, such as the oceans, the atmosphere or rivers. The so-called Shallow Water equations (Saint-Venant equations in the French literature), raise quite different mathematical problems; see e.g. Orenga (1995). The so-called Primitive Equations of the atmosphere and the oceans are the fundamental equations of meteorology and oceanography (height small compared to the radius of the earth): they consist of the NSE with Corriolis force for the horizontal velocity and, in the vertical direction, the conservation of momentum equation is replaced by the hydrostatic equation; coupling with temperature and humidity or salinity equations can be added. The mathematical study of these equations has been recently developed by Lions, Temam and Wang (1992a,b, 1995): after a proper modelling the equations were written in a form similar to the functional form of the NSE but the nonlinear term is more involved and the results are weaker than for the 3D NSE of incompressible fluids. For some other mathematical problems related to the equations of climate see e.g. Beale (1994), Embid and Majda (1996).

In a different context one may also consider the 2D NSE on the sphere with or without Coriolis force. After a proper geometrical setting the results on well-posedness are the same as for 2D-NSE, fairly complete (see Ebin and Marsden (1970) for the Euler equations, and Ghidaglia (1986)).

Another topic related to the Euler equations is the propagation of waves in shallow waters. Suitable asymptotic expansions lead to the Korteweg de Vries equations or the nonlinear Schrödinger equation or related water waves equations. This is a different subject, related to solitons for which the reader is refered to the specialized literature.

Finally NSE are based on the assumption of Newtonian flows. For non Newtonian fluids, one may introduce hyperviscosity ($\varepsilon(-\Delta)^r$ replacing $-\nu\Delta, r > 1$), see Lions (1959, 1965) or nonlinear viscosity (see Ladyzhenskaya (1959)). There is also a large distinct literature on non Newtonian flows applying for instance to polymers; see e.g. the book of Joseph and Renardy (1993) and the references therein.

More Specific Problems. Qualitative Properties

We mention here some other aspects of NSE leading to specific problems and techniques.

First we recall the previously mentioned specific problems of stationary solutions in *unbounded domains* for incompressible and compressible fluids. For incompressible flows in bounded domains nonuniqueness of stationary solutions and their stability are part of *bifurcation theory* using among others the topological degree theory: in general a characteristic parameter is selected (Reynolds number or a similar number), and one studies the number of solutions and their stability as the parameter varies; bifurcations involving two parameters or more become very involved. See Rabinowitz (1973) for an abstract result on bifurcation; for NSE see in particular the work of Golubitsky, Iooss, Kirchgassner, Yudovich and Benjamin and Joseph, and their collaborators; the latter have also very nicely combined theory and experiments. See also the book of Iooss and Joseph (1980) and that of Golubitsky and Schaeffer (1985) which addresses the effect of symmetries.

As we saw, for turbulent flows, even if the data are time independent, the permanent regime might be time dependent. The first occurrence of time dependence is the *Hopf bifurcation*: stationary solutions bifurcate into a time periodic one and the corresponding permanent regime is a time periodic flow. Of course Hopf (1943) was motivated by fluid mechanics when he introduced the bifurcation phenomenon bearing his name. In the recent past Hopf (and other) bifurcations for the NSE were studied by Chenciner, Iooss and others.

When a proper nondimensionalization of the NSE is performed $\nu \cong Re^{-1}$ where Re is the Reynolds number of the flow. The Reynolds number is large for physically interesting flows, e.g. 10^4 to 10^8 for common or industrial flows, $Re = 10^{12}$ to 10^{18} for geophysical flows (on earth or other planets). For Re large, for wall bounded flows, important physical phenomena not well-understood, occur in a thin region near the wall, usually of thickness $Re^{-1/2}$ called the *boundary layer*. We already mentioned the problem still open of convergence (or nonconvergence?) of the Navier-Stokes equations to the Euler equations, for wall bounded flows, as $\nu \to 0$; the difficulty is related of course to insufficient understanding of the boundary layer phenomena.

It is noteworthy to mention that some insight in the qualitative behavior of the solutions to the NSE has been gained from experimental (laboratory) studies during the past decades, they helped and can still help produce mathematical conjectures: see e.g. the experiments by the physicists, Libchaber on boundary layers and bifurcation, by H. Swinney on bifurcations and transition to chaos, and the experiments by the fluid mechanics engineers on bifurcation, B. Benjamin and D. Joseph who, as we already said, combined theory and experiment.

Numerical Analysis and Computational Fluid Dynamics

Due to the complexity of the NSE it is hopeless to find exact solutions for the boundary value problems attached to them except for very specific problems when many symmetries are present. Hence for any problem of practical interest in physics or engineering, one can only hope to compute an approximate solution by numerical simulation. Much effort has been and is currently devoted to this in the mechanical engineering and mathematical communities, not to mention the considerable efforts in geophysical fluid dynamics. At this time the numerical simulation of the NSE themselves (also called Direct Numerical Simulation) cannot be achieved for physically significant flows and some kind of modelling of turbulence is usually used: $k-\varepsilon$ models, Large Eddy Simulations, Smagorinsky model. For DNS in 3D we know, from the conventional theory of turbulence corroborated by the studies on dimension of attractors that $Re^{9/4}$ unknowns must be computed at each time step for a full description of the flow. This is out of reach today but by extrapolating the present improvements in computing power and memory facilities, one may realistically hope to be able to compute significant flows by DNS within one or two decades.

Another motivation of numerical simulations of NSE and Euler equations lies on the mathematical side. Keeping in mind the role played by numerical simulations in the discovery of solitons and the study of the Korteweg de Vries equations, one may hope to obtain, by numerical simulations, some insight on the occurrence of singularities in the NS and/or Euler equations. Of course the problem is considerably more difficult than for KdV since we go from 1D to 3D. So far these attempts have not been successful, due to the limitations on the computing power, but for the same reasons as before there is hope to obtain better insight with the rapid increase of computing power.

As we said in the Introduction there are problems of mathematical substance not all resolved related to the numerical analysis of NSE; in fact it is well-known that any scheme not supported by a good physical insight or a sound mathematical study is doomed to failure; for instance conservation of energy and numerical viscosity are recurrent questions. There is need also to construct more efficient schemes to increase the capacity of the computers at handling larger problems.

Numerical simulations of flows were present, as we said, in the late 40's since the first appearances of computers. The numerical analysis of the incompressible NSE themselves started in the early to mid sixties, conducted by Lax, Lions, Marchuk, Yanenko, Chorin, Temam. It has been and continues to be very active. As we said, the CFD literature is now considerable. A few books close to the mathematical point of view are those of Bernardi and Maday (1992), Brezzi and Fortin (1991), Canuto, Hussaini, Quarteroni and Zang (1987), Chorin (1994), Girault and Raviart (1986), Gottlieb and Orszag (1977), Gresho and Sani (1996) Gunzburger (1989), Heywood, Masuda, Rautman and Solonnikov (Eds) (1991), Temam (1977), Yanenko (1971). See also the articles in the Handbook of Numerical Analysis edited by Ciarlet and Lions (1998).

Control of Flows

A certain activity has developed, especially during the 1990's, addressing the control of flows; incompressible viscous or inviscid flows have been considered.

We consider the NSE in the form (5)-(6) (with $u = g \not\equiv 0$ on $\partial\Omega$ for (6)), and we assume that f and g are under our control: we want to choose best f and/or g so that u and/or p achieve certain goals. The mathematical theory of control of partial differential equations has been developed by Lions (1971). For the NS equations two classes of problems have been recently investigated:

(i) Optimal and Robust Control of Turbulence

(ii) Controllability (NS and Euler equations).

For (ii) the problems are theoretical (mathematical); they are theoretical and computational for (i).

For (i) the purpose of optimal control is to reduce turbulence as measured by a suitable quantity (e.g. L^2-norm of the curl vector or drag); this is a calculus of variation problem. Various forms of this problem have been addressed by a number of authors; see e.g. the book edited by M. Gunzburger (1989). The robust control is a problem of control in the presence of adverse perturbations; see Bewley, Moin and Temam (1997); see also the references therein for the abundant computational literature on control.

The problem of controllability is how to conduct the flow to a certain desired state by acting on the controls (it could be f only on part of Ω or g only on part of $\partial\Omega$). The problem relates to unique continuation results (Mizohata (1958)) and Carleman inequalities and to the description of the image set $\{u(T)\}$, when f, g and possibly u_0 vary in suitable sets: the first results, description of the problems and conjectures appear in Lions (1988, 1990). In a series of articles, Fursikov (1995), Fursikov and Imanuvilov (1994, 1996a,b) prove local exact controllability through f or g, for the linearized and for the NS equations. Coron and Fursikov (1996) proved the global exact controllability of the 2D Navier-Stokes on a manifold without boundary. Coron (1996a) obtained partial results on the controllability of the 2D NSE with Dirichlet boundary conditions. Coron (1996b) proves the approximate boundary (i.e. via g) and the distributed (via f) controllability for 2D incompressible Euler equations.

Figure 1 below shows the appearance of vortices in the boundary layer for a channel flow (numerical simulation of uncontrolled flow). Figure 2 shows the reduction of the drag to nearly its absolute minimum (that of a laminar flow) in a controlled channel flow (from Bewley, Moin and Temam (1997)); the corresponding calculations involve 17 millions of spatial modes.

Figure 1. Formation of coherent structures in the boundary layer for an uncontrolled channel flow.

FIGURE 2. Time evolution of the drag for different values of a parameter T in a controlled flow.

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