The Arithmetic-Geometric Mean Inequality in Precalculus

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Arithmetic-Geometric Mean Inequality

If $a_1, a_2, a_3, \ldots, a_n \geq 0$, then

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 a_3 \cdots a_n}$$

with equality if and only if $a_1 = a_2 = a_3 = \cdots = a_n$.

$$\frac{a+b}{2} \geq \sqrt{ab}$$

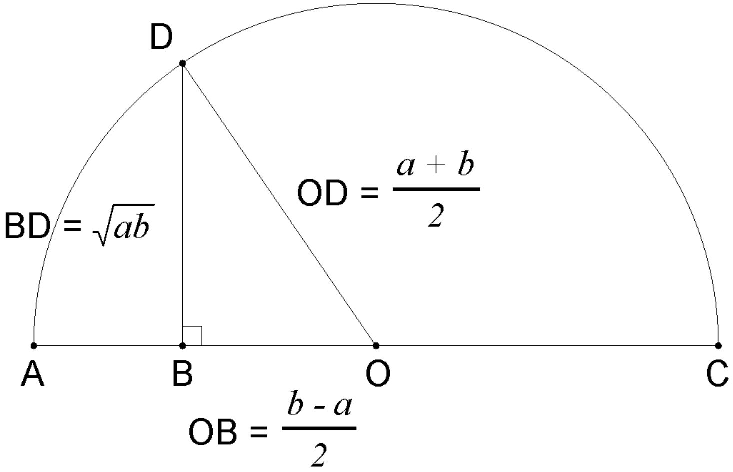


Figure: $AB = a$ and $BC = b$

Generalization of AM-GM Inequality

If $x_1, x_2, x_3, \ldots, x_n > 0$, $w_1, w_2, w_3, \ldots, w_n > 0$, and $w_1 + w_2 + w_3 + \cdots + w_n = 1$, then

$$x_1^{w_1} \cdot x_2^{w_2} \cdot x_3^{w_3} \cdots x_n^{w_n} \leq w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n$$

with equality if and only if $x_1 = x_2 = x_3 = \cdots = x_n$.

Some Optimization Problems

1. Find the rectangle of perimeter P that encloses the largest area.
2. Find the dimensions of the closed box of volume V that requires the least amount of material to build.
3. Find the dimensions of the closed cylindrical can of volume V that requires the least amount of material to build.
4. Find the sides of the triangle of perimeter P that encloses the largest area.
5. Find the cylinder of maximum volume that can be placed inside a cone of height H and base radius R.
6. Find the dimensions of the rectangle of largest area which can be situated so that two adjacent sides lie on the positive coordinate axes and the remaining vertex lies on the line $2x + 3y = 12$.
7. Find the largest possible volume of a right circular cylinder inscribed in a sphere of radius R.

Problem 1: Find the rectangle of perimeter P that encloses the largest area.

Let x and y be the length and width of the rectangle, respectively. Maximize $A = xy$ subject to

$$2x + 2y = P \quad \Rightarrow \quad x + y = \frac{P}{2}.$$

By the Arithmetic-Geometric Mean Inequality,

$$\begin{aligned} A &= xy \\ &\leq \left(\frac{x+y}{2}\right)^2 \\ &= \left(\frac{P}{4}\right)^2 \\ &= \frac{P^2}{16}, \end{aligned}$$

with equality if and only if $x = y = \dfrac{P}{4}$.

Problem 2: Find the dimensions of the closed box of volume V that requires the least amount of material to build.

Let x, y, and z be the length, width, and height of the closed box, respectively. Minimize $S.A. = 2xy + 2xz + 2yz$ subject to $xyz = V$.

By the Arithmetic-Geometric Mean Inequality,

$$\begin{aligned} S.A. &= 2xy + 2xz + 2yz \\ &\geq 3\sqrt[3]{(2xy)(2xz)(2yz)} \\ &= 6\sqrt[3]{x^2y^2z^2} \\ &= 6(xyz)^{2/3} \\ &= 6V^{2/3}, \end{aligned}$$

with equality if and only if $x = y = z = V^{1/3}$.

Problem 3: Find the dimensions of the closed cylindrical can of volume V that requires the least amount of material to build.

Let r and h be the radius and the height of the cylindrical can, respectively. Minimize $S.A. = 2\pi r^2 + 2\pi rh$ subject to $\pi r^2 h = V$.

By the Arithmetic-Geometric Mean Inequality,

$$\begin{aligned} S.A. &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + \pi rh + \pi rh \\ &\geq 3\sqrt[3]{(2\pi r^2)(\pi rh)(\pi rh)} \\ &= 3\sqrt[3]{2\pi(\pi r^2 h)^2} \\ &= 3\sqrt[3]{2\pi V^2} \\ &= 3\sqrt[3]{2\pi}V^{2/3}, \end{aligned}$$

with equality if and only if $h = 2r$, or equivalently $V = 2\pi r^3$. Thus, $r = \sqrt[3]{\frac{V}{2\pi}}$ and $h = 2\sqrt[3]{\frac{V}{2\pi}}$.

Problem 4: Find the sides of the triangle of perimeter P that encloses the largest area.

Let a, b, and c be the length of the three sides of the triangle, respectively. Maximize $A = \sqrt{s(s-a)(s-b)(s-c)}$ subject to $a+b+c = P$ and $s = \frac{1}{2}P$.

By the Arithmetic-Geometric Mean Inequality,

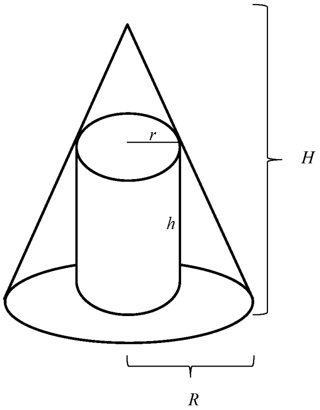
$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{\tfrac{1}{2}P(\tfrac{1}{2}P-a)(\tfrac{1}{2}P-b)(\tfrac{1}{2}P-c)} \\ &\leq \sqrt{\tfrac{1}{2}P\left(\frac{(\frac{1}{2}P-a)+(\frac{1}{2}P-b)+(\frac{1}{2}P-c)}{3}\right)^3} \\ &= \sqrt{\frac{(\frac{1}{2}P)^4}{27}} \\ &= \frac{P^2}{12\sqrt{3}}, \end{aligned}$$

with equality if and only if $a = b = c = \frac{P}{3}$.

Problem 5: Find the cylinder of maximum volume that can be placed inside a cone of height H and base radius R.

Maximize $V_{cyl} = \pi h^2 h$ subject to $\frac{1}{3}\pi R^2 H = V_{cone}$.

Using similar triangles, $h = \frac{H(R-r)}{R}$.



By the Arithmetic-Geometric Mean Inequality,

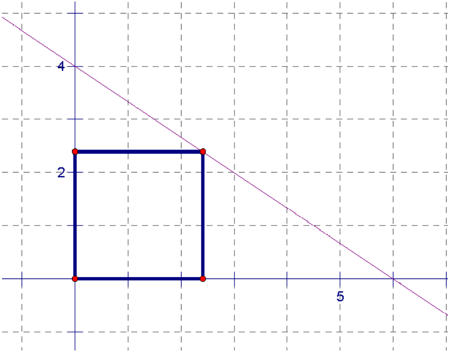
$$\begin{aligned}
V_{cyl} &= \pi r^2 h \\
&= \pi r^2 \left(\frac{H(R-r)}{R} \right) \\
&= \frac{\pi H}{2R} (2r^2(R-r)) \\
&= \frac{\pi H}{2R} (r(r)(2R-2r)) \\
&\leq \frac{\pi H}{2R} \left(\frac{r+r+(2R-2r)}{3} \right)^3 \\
&= \frac{\pi H}{2R} \left(\frac{2R}{3} \right)^3 \\
&= \frac{4}{9} V_{cone},
\end{aligned}$$

with equality if and only if $r = 2R - 2r$, or equivalently $r = \frac{2R}{3}$ and $h = \frac{H}{3}$.

Problem 6: Find the dimensions of the rectangle of largest area which can be situated so that two adjacent sides lie on the positive coordinate axes and the remaining vertex lies on the line $2x + 3y = 12$.

Maximize $A = xy$ subject to $2x + 3y = 12$.

Using similar triangles, $\frac{4}{6} = \frac{y}{6-x}$.



By the Arithmetic-Geometric Mean Inequality,

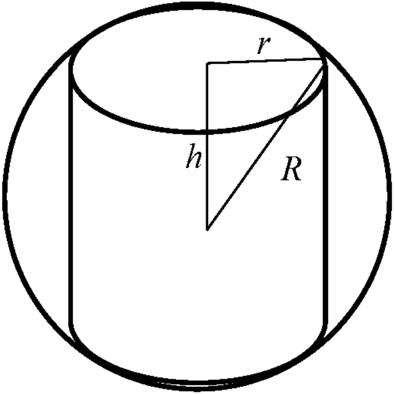
$$
\begin{aligned}
A &= xy \\
&= \frac{4}{6}x(6-x) \\
&\leq \frac{4}{6}\left(\frac{x+(6-x)}{2}\right)^2 \\
&= 6,
\end{aligned}
$$

with equality if and only if $x = 6 - x$, or equivalently $x = 3$ and $y = 2$.

Problem 7: Find the largest possible volume of a right circular cylinder inscribed in a sphere of radius R.

Maximize $V_{cyl} = \pi r^2 h$ subject to $\frac{4}{3}\pi R^3 = V_s$.

Using the Pythagorean Theorem, $r^2 = R^2 - h^2$



By the Arithmetic-Geometric Mean Inequality,

$$
\begin{aligned}
(V_{cyl})^2 &= (\pi r^2(2h))^2 \\
&= 4\pi^2 h^2 (R^2 - h^2)^2 \\
&= 2\pi^2 (2h^2)(R^2 - h^2)(R^2 - h^2) \\
&\leq 2\pi^2 \left(\frac{2h^2 + (R^2 - h^2) + (R^2 - h^2)}{3} \right)^3 \\
&= 2\pi^2 \left(\frac{2R^2}{3} \right)^3 \\
&= \frac{1}{3} (\frac{4}{3}\pi R^3)^2 \\
&= \frac{1}{3} V_s^2, \\
\Rightarrow V_{cyl} &\leq \frac{1}{\sqrt{3}} V_s,
\end{aligned}
$$

with equality if and only if $h = \frac{R}{\sqrt{3}}$ and $r = \frac{\sqrt{6}}{3} R$.