

LECTURE 6

ILLUSTRATIVE PROBLEMS:

Let us discuss few representative problems dealing with complex state of stress to be solved either analytically or graphically.

PROB 1: A circular bar 40 mm diameter carries an axial tensile load of 105 kN. What is the Value of shear stress on the planes on which the normal stress has a value of 50 MN/m² tensile.

Solution:

$$\begin{aligned} \text{Tensile stress } \sigma_y &= F / A = 105 \times 10^3 / \pi \times (0.02)^2 \\ &= 83.55 \text{ MN/m}^2 \end{aligned}$$

Now the normal stress on an oblique plane is given by the relation

$$\sigma_\theta = \sigma_y \sin^2 \theta$$

$$50 \times 10^6 = 83.55 \text{ MN/m}^2 \times 10^6 \sin^2 \theta$$

$$\theta = 50^\circ 68'$$

The shear stress on the oblique plane is then given by

$$\begin{aligned} \tau_\theta &= 1/2 \sigma_y \sin 2\theta \\ &= 1/2 \times 83.55 \times 10^6 \times \sin 101.36 \\ &= 40.96 \text{ MN/m}^2 \end{aligned}$$

Therefore the required shear stress is 40.96 MN/m²

PROB 2:

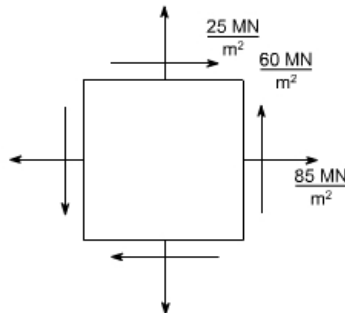
For a given loading conditions the state of stress in the wall of a cylinder is expressed as follows:

- (a) 85 MN/m² tensile
- (b) 25 MN/m² tensile at right angles to (a)
- (c) Shear stresses of 60 MN/m² on the planes on which the stresses (a) and (b) act; the sheer couple acting on planes carrying the 25 MN/m² stress is clockwise in effect.

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged

Solution:

The problem may be attempted both analytically as well as graphically. Let us first obtain the analytical solution



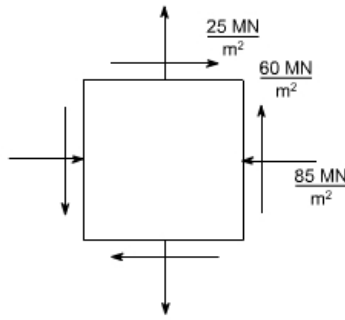
The principle stresses are given by the formula

$$\begin{aligned} \sigma_1 \text{ and } \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \frac{1}{2}(85 + 25) \pm \frac{1}{2}\sqrt{(85 - 25)^2 + (4 \times 60^2)} \\ &= 55 \pm \frac{1}{2} \cdot 60\sqrt{5} = 55 \pm 67 \\ \Rightarrow \sigma_1 &= 122 \text{ MN/m}^2 \\ \sigma_2 &= -12 \text{ MN/m}^2 \text{ (compressive)} \end{aligned}$$

For finding out the planes on which the principle stresses act us the equation $\tan 2\theta = \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right)$

The solution of this equation will yeild two values θ i.e they θ_1 and θ_2 giving $\theta_1 = 31^\circ 71'$ & $\theta_2 = 121^\circ 71'$

(b) In this case only the loading (a) is changed i.e. its direction had been changed. While the other stresses remains unchanged hence now the block diagram becomes.



Again the principal stresses would be given by the equation.

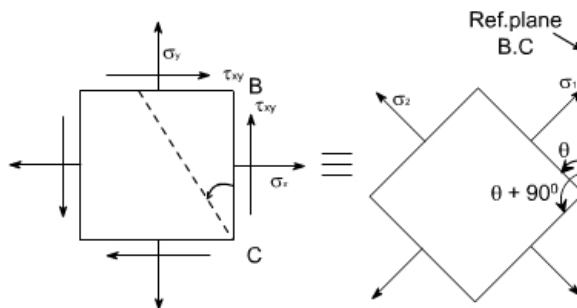
$$\begin{aligned} \sigma_1 \text{ \& } \sigma_2 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \frac{1}{2}(-85 + 25) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)} \\ &= \frac{1}{2}(-60) \pm \frac{1}{2}\sqrt{(-85 - 25)^2 + (4 \times 60^2)} \\ &= -30 \pm \frac{1}{2}\sqrt{12100 + 14400} \\ &= -30 \pm 81.4 \end{aligned}$$

$$\sigma_1 = 51.4 \text{ MN/m}^2; \sigma_2 = -111.4 \text{ MN/m}^2$$

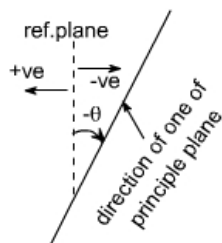
Again for finding out the angles use the following equation.

$$\begin{aligned} \tan 2\theta &= \left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \\ &= \frac{2 \times 60}{-85 - 25} = + \frac{120}{-110} \\ &= -\frac{12}{11} \\ 2\theta &= \tan^{-1}\left(-\frac{12}{11}\right) \\ \Rightarrow \theta &= -23.74^\circ \end{aligned}$$

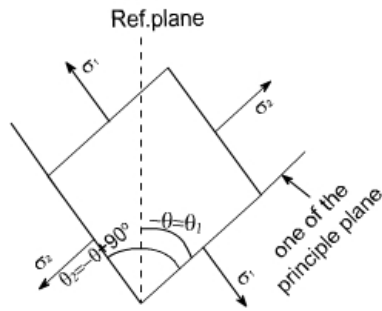
Thus, the two principle stresses acting on the two mutually perpendicular planes i.e principle planes may be depicted on the element as shown below:



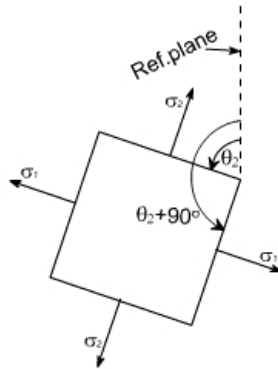
So this is the direction of one principle plane & the principle stresses acting on this would be σ_1 when is acting normal to this plane, now the direction of other principal plane would be $90^\circ + \theta$ because the principal planes are the two mutually perpendicular plane, hence rotate the another plane $\theta + 90^\circ$ in the same direction to get the another plane, now complete the material element if θ is negative that means we are measuring the angles in the opposite direction to the reference plane BC .



Therefore the direction of other principal planes would be $\{-\theta + 90\}$ since the angle $-\theta$ is always less in magnitude then 90 hence the quantity $(-\theta + 90)$ would be positive therefore the inclination of other plane with reference plane would be positive therefore if just complete the Block. It would appear as



If we just want to measure the angles from the reference plane, then rotate this block through 180° so as to have the following appearance.



So whenever one of the angles comes negative to get the positive value,

first Add 90° to the value and again add 90° as in this case $\theta = -23^\circ 74'$

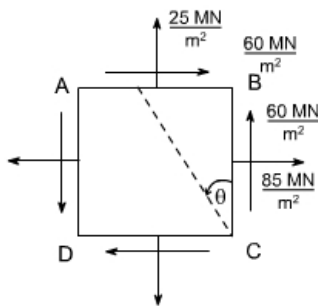
so $\theta_1 = -23^\circ 74' + 90^\circ = 66^\circ 26'$. Again adding 90° also gives the direction of other principle planes

i.e $\theta_2 = 66^\circ 26' + 90^\circ = 156^\circ 26'$

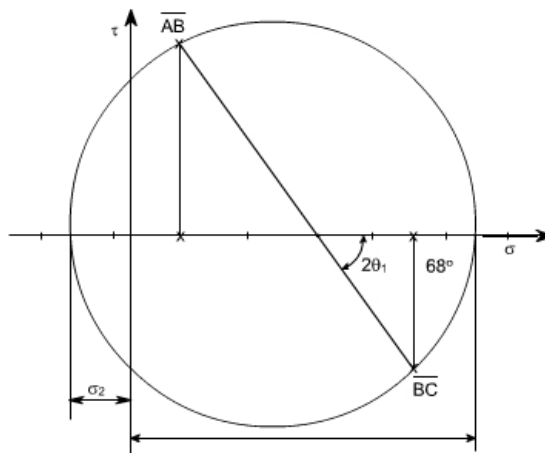
This is how we can show the angular position of these planes clearly.

GRAPHICAL SOLUTION:

Mohr's Circle solution: The same solution can be obtained using the graphical solution i.e the Mohr's stress circle, for the first part, the block diagram becomes



Construct the graphical construction as per the steps given earlier.



Taking the measurements from the Mohr's stress circle, the various quantities computed are

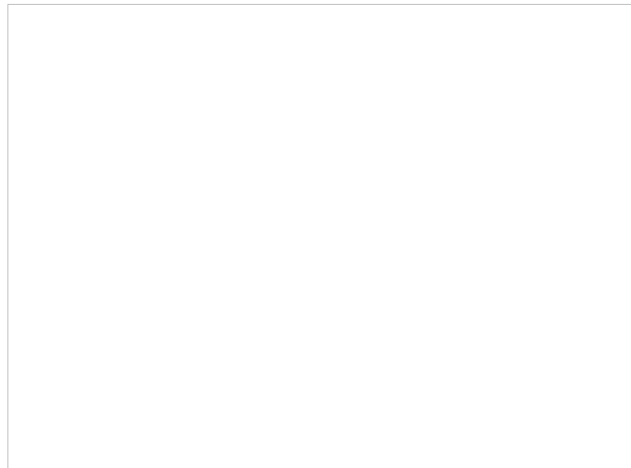
$$\sigma_1 = 120 \text{ MN/m}^2 \text{ tensile}$$

$$\sigma_2 = 10 \text{ MN/m}^2 \text{ compressive}$$

$$\theta_1 = 34^\circ \text{ counter clockwise from BC}$$

$$\theta_2 = 34^\circ + 90 = 124^\circ \text{ counter clockwise from BC}$$

Part Second : The required configuration i.e the block diagram for this case is shown along with the stress circle.



By taking the measurements, the various quantities computed are given as

$$\sigma_1 = 56.5 \text{ MN/m}^2 \text{ tensile}$$

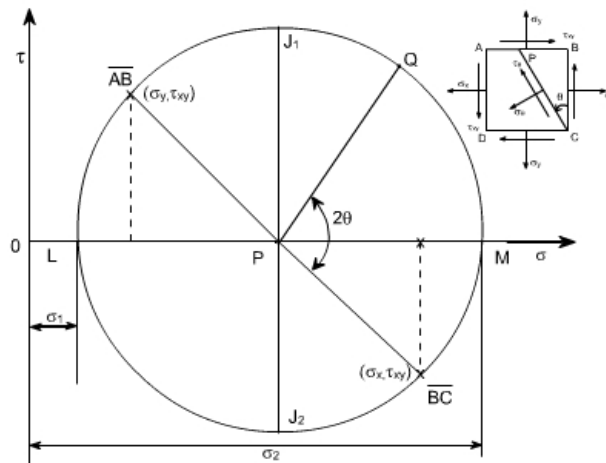
$$\sigma_2 = 106 \text{ MN/m}^2 \text{ compressive}$$

$$\theta_1 = 66^\circ 15' \text{ counter clockwise from BC}$$

$$\theta_2 = 156^\circ 15' \text{ counter clockwise from BC}$$

Salient points of Mohr's stress circle:

1. complementary shear stresses (on planes 90° apart on the circle) are equal in magnitude
2. The principal planes are orthogonal: points L and M are 180° apart on the circle (90° apart in material)
3. There are no shear stresses on principal planes: point L and M lie on normal stress axis.
4. The planes of maximum shear are 45° from the principal points D and E are 90° , measured round the circle from points L and M.
5. The maximum shear stresses are equal in magnitude and given by points D and E
6. The normal stresses on the planes of maximum shear stress are equal i.e. points D and E both have normal stress co-ordinate which is equal to the two principal stresses.



As we know that the circle represents all possible states of normal and shear stress on any plane through a stresses point in a material. Further we have seen that the co-ordinates of the point 'Q' are seen to be the same as those derived from equilibrium of the element. i.e. the normal and shear stress components on any plane passing through the point can be found using Mohr's circle. Worthy of note:

1. The sides AB and BC of the element ABCD, which are 90° apart, are represented on the circle by \overline{AP} and \overline{BP} and they are 180° apart.
2. It has been shown that Mohr's circle represents all possible states at a point. Thus, it can be seen at a point. Thus, it, can be seen that two planes LP and PM, 180° apart on the diagram and therefore 90° apart in the material, on which shear stress τ_θ is zero. These planes are termed as principal planes and normal stresses acting on them are known as principal stresses.

Thus , $\sigma_1 = OL$

$$\sigma_2 = OM$$

3. The maximum shear stress in an element is given by the top and bottom points of the circle i.e by points J_1 and J_2 ,Thus the maximum shear stress would be equal to the radius of i.e. $\tau_{max} = 1/2(\sigma_1 - \sigma_2)$,the corresponding normal stress is obviously the distance $OP = 1/2(\sigma_1 + \sigma_2)$, Further it can also be seen that the planes on

which the shear stress is maximum are situated 90° from the principal planes (on circle), and 45° in the material.

4. The minimum normal stress is just as important as the maximum. The algebraic minimum stress could have a magnitude greater than that of the maximum principal stress if the state of stress were such that the centre of the circle is to the left of origin.

i.e. if $\sigma_1 = 20 \text{ MN/m}^2$ (say)

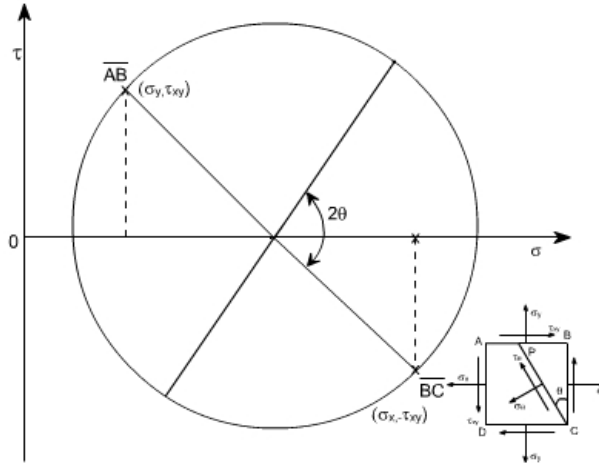
$$\sigma_2 = -80 \text{ MN/m}^2 \text{ (say)}$$

$$\text{Then } \tau_{\max}^m = (\sigma_1 - \sigma_2 / 2) = 50 \text{ MN/m}^2$$

It should be noted that the principal stresses are considered a maximum or minimum mathematically e.g. a compressive or negative stress is less than a positive stress, irrespective of numerical value.

5. Since the stresses on perpendicular faces of any element are given by the co-ordinates of two diametrically opposite points on the circle, thus, the sum of the two normal stresses for any and all orientations of the element is constant, i.e. This sum is an invariant for any particular state of stress.

Sum of the two normal stress components acting on mutually perpendicular planes at a point in a state of plane stress is not affected by the orientation of these planes.



This can be also understood from the circle. Since AB and BC are diametrically opposite thus, whatever may be their orientation, they will always lie on the diameter or we can say that their sum won't change, it can also be seen from analytical relations

$$\text{We know } \sigma_n = \frac{(\sigma_x + \sigma_y)}{2} + \frac{(\sigma_x - \sigma_y)}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

on plane BC; $\theta = 0$

$$\sigma_{n1} = \sigma_x$$

on plane AB; $\theta = 270^\circ$

$$\sigma_{n2} = \sigma_y$$

$$\text{Thus } \sigma_{n1} + \sigma_{n2} = \sigma_x + \sigma_y$$

6. If $\sigma_1 = \sigma_2$, the Mohr's stress circle degenerates into a point and no shearing stresses are developed on xy plane.

7. If $\sigma_x + \sigma_y = 0$, then the center of Mohr's circle coincides with the origin of $\sigma - \tau$ co-ordinates.