LECTURE 25 and 26

Simple Bending Theory OR Theory of Flexure for Initially Straight Beams

(The normal stress due to bending are called flexure stresses)

Preamble:

When a beam having an arbitrary cross section is subjected to a transverse loads the beam will bend. In addition to bending the other effects such as twisting and buckling may occur, and to investigate a problem that includes all the combined effects of bending, twisting and buckling could become a complicated one. Thus we are interested to investigate the bending effects alone, in order to do so, we have to put certain constraints on the geometry of the beam and the manner of loading.

Assumptions:

The constraints put on the geometry would form the **assumptions:**

1. Beam is initially **straight** , and has a **constant cross-section.**
2. Beam is made of **homogeneous material** and the beam has a **longitudinal plane of symmetry.**
3. Resultant of the applied loads lies in the plane of symmetry.
4. The geometry of the overall member is such that bending not buckling is the primary cause of failure.
5. Elastic limit is nowhere exceeded and **'E'** is same in tension and compression.
6. Plane cross - sections remains plane before and after bending.

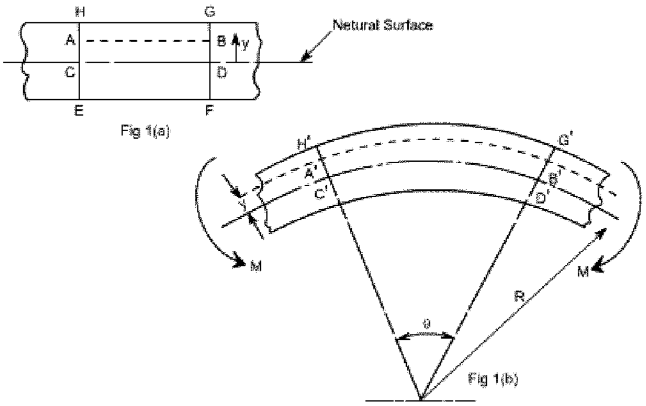


Fig 1(a)

Fig 1(b)

Let us consider a beam initially unstressed as shown in fig 1(a). Now the beam is subjected to a constant bending moment (i.e. 'Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b)

As a result of this bending, the top fibers of the beam will be subjected to tension and the bottom to compression it is reasonable to suppose, therefore, **that some where between the two there are points at which the stress is zero. The locus of all such points is known as neutral axis** . The radius of curvature R is then measured to this axis. For symmetrical sections the N. A. is the axis of symmetry but what ever the section N. A. will always pass through the centre of the area or centroid.

The above restrictions have been taken so as to eliminate the possibility of 'twisting' of the beam.

Concept of pure bending:

Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

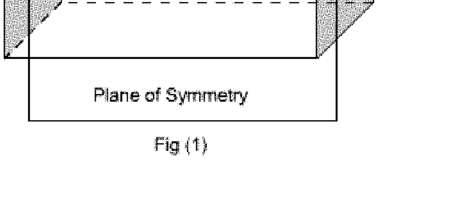


Fig (1)

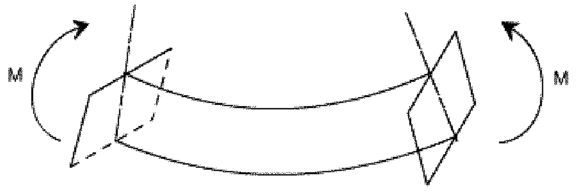
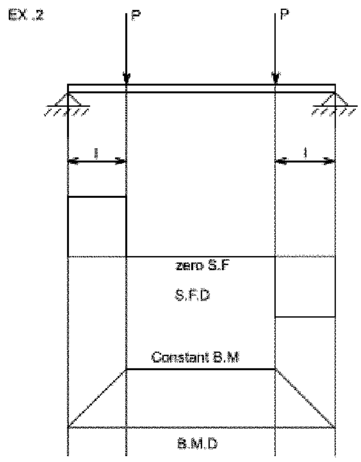
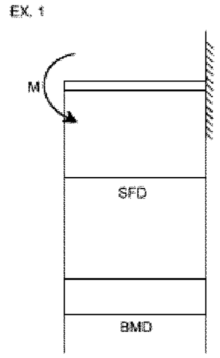


Fig (2)

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :

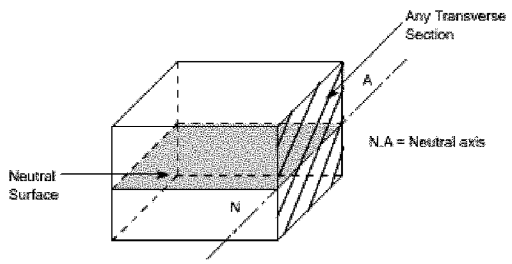




When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.

2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axisNeutral axis **(N A)** .

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle θ.

Consider now fiber AB in the material, at adistance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre } AB = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

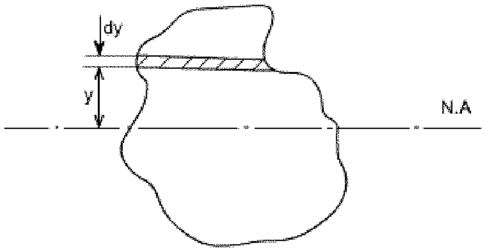
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance 'y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R} y$$

if the shaded strip is of area 'dA'

then the force on the strip is

$$F = \sigma\,\delta A = \frac{E}{R} y\,\delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R} y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R} y^2\,\delta A = \frac{E}{R} \sum y^2 \delta A$$

Now the term $\sum y^2 \delta A$ is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max^m} = \frac{M}{T} y_{max^m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max^m}} \sigma_{max^m}$$

For ready comparison of the strength of various beam cross-section this relationship is some times written in the form

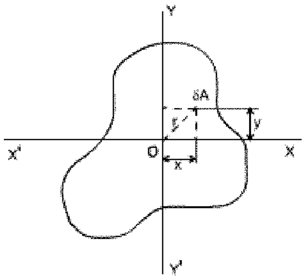
$$M = Z\,\sigma_{max^m} \quad \text{where } Z = \frac{I}{y_{max^m}}$$ **Is termed as section modulus**

The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

Theorems to determine second moment of area: There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area :

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance or $\int y^2 \, dA$.



Consider any cross-section having small element of area d A then by the definition

I_x(Mass Moment of Inertia about x-axis) = $\int y^2 \, dA$ and I_y(Mass Moment of Inertia about y-axis) = $\int x^2 \, dA$

Now the moment of inertia about an axis through 'O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

$$
\begin{aligned}
&= \int r^2 dA \\
&= \int (x^2 + y^2) dA \\
&= \int x^2 dA + \int y^2 dA \\
&= I_X + I_Y
\end{aligned}
$$

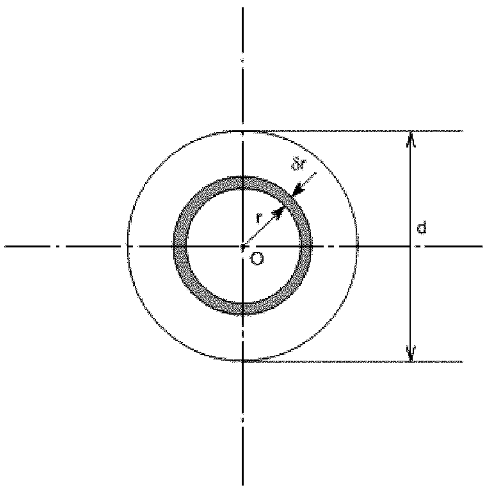
$$\text{or } J = I_X + I_Y \qquad \text{..........} (1)$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness δr located at a radius 'r'.

Than the area of the circular strip would be dA = 2πr. δr

Thus $J = \int r^2 dA$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 2\pi r \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \delta r$$

$$J = 2\pi \left[\frac{r^4}{4}\right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however,by perpendicular axistheorem

$J = I_x + I_y$

But for the circular cross-section ,the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

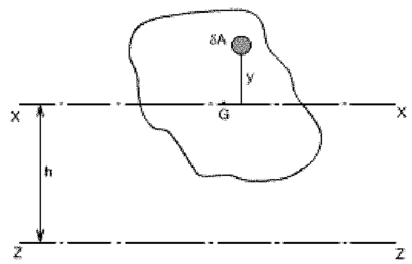
for a hollow circular sectionof diameter D and d, the values of J and I are defined as

$$J = \frac{\pi(D^4 - d^4)}{32}$$

$$I = \frac{\pi(D^4 - d^4)}{64}$$

Parallel Axis Theorem:

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.



If 'ZZ' is any axis in the plane of cross-section and 'XX' is a parallel axis through the centroid G, of the cross-section, then

$I_z = \int (y + h)^2\, dA$ by definition (moment of inertia about an axis ZZ)

$$= \int (\ + 2yh + h^2)\, dA$$

$$= \int y^2 dA + h^2 \int dA + 2h \int y dA$$

Since $\int y dA = 0$

$$= \int y^2 dA + h^2 \int dA$$

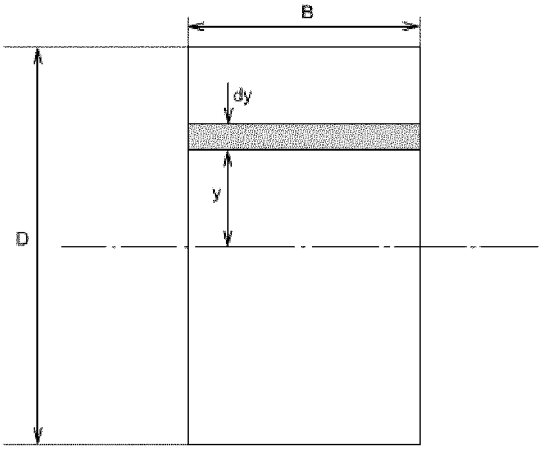
$$= \int y^2 dA + h^2 A$$

$I_z = I_x + Ah^2$ $\quad I_x = I_G$ (since cross-section axes also pass through G)

Where A = Total area of the section

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below :



Consider the rectangular beam cross-section as shown above and an element of area **dA**, thickness **dy**, breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 dA$$

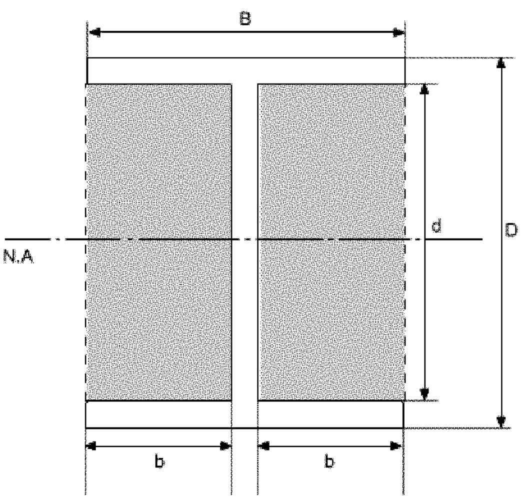
Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$\begin{aligned} I_{N.A} &= \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 (B\, dy) \\ &= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 dy \\ &= B\left[\frac{y^3}{3}\right]_{\frac{D}{2}}^{\frac{-D}{2}} \\ &= \frac{B}{3}\left[\frac{D^3}{8} - \left(\frac{-D^3}{8}\right)\right] \\ &= \frac{B}{3}\left[\frac{D^3}{8} + \frac{D^3}{8}\right] \\ I_{N.A} &= \frac{BD^3}{12} \end{aligned}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of **0** to **D**.

Therefore $I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$

These standards formulas prove very convenient in the determination of I_{NA} for build up sections which can be conveniently divided into rectangles. For instance if we just want to find out the Moment of Inertia of an I - section, then we can use the above relation.



$$I_{N.A} = I_{\text{of dotted rectangle}} - I_{\text{of shaded portion}}$$

$$\therefore I_{N.A} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

$$I_{N.A} = \frac{BD^3}{12} - \frac{bd^3}{6}$$

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